PHYSICAL REVIEW LETTERS

VOLUME 67 22 JULY 1991 NUMBER 4

Breathing Vortex Solitons in Nonrelativistic Chern-Simons Gauge Theory

Z. F. Ezawa and M. Hotta
Department of Physics, Tohoku University, Sendai 980, Japan

A. Iwazaki
Department of Physics, Nishogakusha University, Ohi 2590, Shonan-machi 277, Japan
(Received 1 March 1991)

We present exact solutions for breather solitons in the (2+1)-dimensional Chern-Simons gauge theory with external magnetic field B. The breather soliton is a time-dependent solution whose size is oscillating and whose center of mass moves in a cyclotron motion. These solitons describe how the Jackiw-Pi solitons behave in the external magnetic field. We then apply the Bohr-Sommerfeld semiclassical quantization to the breathing mode and the cyclotron motion. It is found that the orbital angular momentum takes on an integer value and that the quantum soliton is in the Landau level.

PACS numbers: 03.65.Ge, 11.10.Lm, 11.15.−q, 11.17.+y

Recently much attention has been paid to vortex solitons in the (2+1)-dimensional Chern-Simons (CS) gauge theories [1-6]. These vortex solitons are anyons [7], and play a crucial role in the fractional quantum Hall effect (FQHE) [8] and in anyon superconductivity [9]. See also Refs. [10] and [11].

In order to study the solitons in detail it is useful to have exact solutions. Jackiw and Pi (JP) [3] have found analytic solutions of the vortex solitons in a nonrelativistic CS gauge theory. The solutions are obtained by solving a simple self-dual equation derived by taking a limit of the critical coupling constant just as in the case of the Prasad-Sommerfield monopoles [12]. Their model does not involve an external magnetic field, although this is an essential ingredient for its application to the FQHE.

In this paper we analyze their model by including an external magnetic field, and present analytic *time-dependent* solutions of vortex solitons. These solutions are reduced to the JP solutions in the limit of vanishing external magnetic field. Therefore, our solutions represent how the JP solitons behave in the external magnetic field. As is expected, they make a cyclotron motion. We also reveal a *breathing* mode of the solitons, i.e., the oscillation of the length scale of the solitons. This breathing is an internal motion of the solitons induced by the external magnetic field. We then apply a semiclassical quantization to the breathing mode and the cyclotron motion. It is found that the orbital angular momentum takes an integer value and that the soliton is in the Landau levels. As far as we know, our exact solutions are the first example in which these features are discussed for quantum solitons.

We consider a (2+1)-dimensional nonrelativistic CS gauge theory with a uniform external magnetic field B. The system consists of the CS gauge field a_μ and the Schrödinger field ψ coupled with it. The action reads

$$S=\int d^3x\,\mathcal{L}\,, \tag{1}$$

with

$$\mathcal{L}=\psi^\dagger(i\partial_0+a_0)\psi-\frac{1}{2m}\psi^\dagger(i\partial_k+a_k-eA_k)^2\psi+\frac{g}{2}(\psi^\dagger\psi)^2-\frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda\,. \tag{2}$$

The latin index runs over 1,2 and the greek index over 0,1,2. Here, A_k is the external magnetic potential such that

$$A_i=-(B/2)\varepsilon_{ij}x_j\,, \tag{3}$$

and α is the statistics parameter. We have chosen the electromagnetic coupling constant of ψ to be $-e$ and the CS coupling constant to be unity. When $B=0$, the action becomes the one which JP analyzed.

It is a special feature of this model that there is an SO(2,1) symmetry [4] in the absence of the external magnetic field. The generators are the Hamiltonian H, the dilation generator D, and the special conformal generator K. In particular, when $B=0$, the action (1) is invariant under dilation:

$$\begin{aligned} &\mathbf{x}\to\mathbf{x}'=\Omega^{-1}\mathbf{x},\quad t\to t'=\Omega^{-2}t\,,\\ &\psi\to\psi'=\Omega\psi(t,x)\,,\\ &a_k\to a_k'=\Omega a_k,\quad a_0\to a_0'=\Omega^2 a_0\,, \end{aligned} \tag{4}$$

where Ω is a constant. When the external magnetic field is switched on, only the Hamiltonian remains a conserved quantity. It is interesting to analyze explicitly the relation between the external magnetic field and the breakdown of dilation invariance, which helps us to construct soliton solutions in the presence of the external magnetic field.

For this purpose we consider a *time-dependent* dilation [13]:

$$\begin{aligned} &\mathbf{x}\to\mathbf{x}'=\Omega(t)^{-1}\mathbf{x},\quad t\to t'=\int_0^t d\tau\,\Omega(\tau)^{-2}\,,\\ &\psi\to\psi'(t',\mathbf{x}')=\Omega(t)\exp\left[-i\frac{mr^2}{2}\frac{\partial_t\Omega(t)}{\Omega(t)}\right]\psi(t,\mathbf{x})\,,\\ &a_\mu\to a_\mu'(t',\mathbf{x}')=\frac{\partial x^\nu}{\partial x'^\mu}a_\nu(t,\mathbf{x})\,, \end{aligned} \tag{5}$$

which reduces to (4) for constant Ω. Because this is not a symmetry of the system, its application leads to a modification of the dynamics. When we choose

$$\Omega(t)=\cos(\omega t/2),\quad \omega=eB/m\,, \tag{6}$$

it turns out that the harmonic-potential term $\psi^\dagger(e^2A_k^2/2m)\psi$ is removed from the Lagrangian density. We next consider the *time-dependent* rotation:

$$\begin{aligned} &t'\to t''=t'\,,\\ &\mathbf{x}'\to\mathbf{x}''=\begin{pmatrix}\cos(\omega t/2) & \sin(\omega t/2)\\ -\sin(\omega t/2) & \cos(\omega t/2)\end{pmatrix}\mathbf{x}'. \end{aligned} \tag{7}$$

The effect of this is to remove the term $\psi'^\dagger(eA_k'/2m)(i\partial'^k+a_k')\psi'$ from the Lagrangian density. The combined transformation is

$$\begin{aligned} &t''=\frac{2}{\omega}\tan(\omega t/2)\,,\quad \mathbf{x}''=\begin{pmatrix}1 & \tan(\omega t/2)\\ -\tan(\omega t/2) & 1\end{pmatrix}\mathbf{x}\,,\\ &\psi''(t'',\mathbf{x}'')=\cos(\omega t/2)\exp\left[i\frac{m\omega}{4}r^2\tan(\omega t/2)\right]\psi(t,\mathbf{x})\,,\\ &a_\mu''(t'',\mathbf{x}'')=\frac{\partial x^\nu}{\partial x''^\mu}a_\nu(t,\mathbf{x})\,. \end{aligned} \tag{8}$$

This is a unitary transformation since $\int d^2x\,\psi^\dagger\psi$ is invariant. It is straightforward to show that after this transformation the action reads

$$S=\int d^3x''\mathcal{L}''\,, \tag{9}$$

with

$$\mathcal{L}''=\psi''^\dagger(i\partial_0''+a_0'')\psi''-\frac{1}{2m}\psi''^\dagger(i\partial_k''+a_k'')^2\psi''+\frac{g}{2}(\psi''^\dagger\psi'')^2-\frac{1}{4\alpha}\varepsilon^{\mu\nu\lambda}a_\mu''\partial_\nu a_\lambda''\,. \tag{10}$$

We conclude that the system with external magnetic field B is related to the system without B by the transformation (8). Consequently, we are able to create solutions of the system with B from those without B by using the above transformation.

When the coupling constant g takes a critical value g_c with

$$g_c=2|\alpha|/m\,, \tag{11}$$

the action (9) with (10) gives rise to a self-dual equation, which is the Liouville equation [3]. The JP soliton solutions are given by

$$\psi(\mathbf{x})=\exp\left[i\frac{\alpha}{|\alpha|}(N-1)\theta(\mathbf{r})\right]\times\left(\frac{2}{|\alpha|}\right)^{1/2}\frac{N}{r}\left[\left(\frac{r}{r_0}\right)^N+\left(\frac{r_0}{r}\right)^N\right]^{-1}, \tag{12}$$

and

$$a_i(\mathbf{x})=-2N\frac{\alpha}{|\alpha|}\frac{\varepsilon_{ij}r_j}{r^2}\left[1+\left(\frac{r_0}{r}\right)^{2N}\right]^{-1}, \tag{13}$$

where $\mathbf{r}=\mathbf{x}-2\mathbf{R}$, $\theta(\mathbf{r})=\tan(r^2/r^1)$, and $N=2,3,\ldots$. Here, $\mathbf{R}$ and r_0 are free parameters. The center of the soliton is placed at $2\mathbf{R}$ and its scale is fixed by r_0. It is obvious that the energy of these solitons does not depend on r_0 and R because of the dilation symmetry and the translation symmetry.

The electric charge eQ of this soliton is given by

$$eQ=e\int d^2x\,\psi^\dagger\psi=\frac{2\pi N}{|\alpha|}e\,. \tag{14}$$

The spin S of the soliton is defined by the total angular momentum J of the static soliton, i.e., $S=J$ with

$$J=\int d^2x\,\varepsilon_{ij}x_i\mathcal{P}_j\,, \tag{15}$$

where

$$\mathcal{P}_j=(1/2i)[\psi^\dagger(\partial_j-ia_j)\psi-(\partial_j+ia_j)\psi^\dagger\psi]\,. \tag{16}$$

Hence, using (12) and (13) we find that

$$S=-2\pi N/\alpha\,. \tag{17}$$

The mass M of the soliton is determined as follows. By performing a Galilean boost, we obtain a vortex soliton moving with velocity V. Evaluating the energy we get that $E=\frac{1}{2}MV^2$ with

$$M=mQ=m\int d^2x\,\psi^\dagger\psi=\frac{2\pi N}{|\alpha|}m\,. \tag{18}$$

Thus, the charge-mass ratio eQ/M of the soliton is equal to e/m.

We now present time-dependent soliton solutions, which are generated by performing the transformation

(8) to the static solutions (12) and (13):

$$\psi(t,\mathbf{x}) = \exp\left[-i\left(\frac{m\omega}{4}\mathbf{x}^2\tan(\omega t/2) + \frac{\alpha}{|\alpha|}\frac{\omega t}{2}(N-1)\right)\right]\exp\left(i\frac{\alpha}{|\alpha|}(N-1)\theta(\mathbf{r}(t))\right)$$
$$\times\left(\frac{2}{|\alpha|}\right)^{1/2}\frac{N}{r(t)}\left(\left|\frac{r(t)}{r_0\cos(\omega t/2)}\right|^N + \left|\frac{r_0\cos(\omega t/2)}{r(t)}\right|^N\right)^{-1} \tag{19}$$

and

$$a_i(t,\mathbf{x}) = -\frac{\alpha}{|\alpha|}\frac{\varepsilon_{ij}r_j(t)}{r(t)^2}2N\left(1+\left|\frac{r_0\cos(\omega t/2)}{r(t)}\right|^{2N}\right)^{-1}, \tag{20}$$

where $\mathbf{r}(t) = \mathbf{x} - 2\mathbf{R}(t)$ with

$$\mathbf{R}(t) = \cos(\omega t/2)\begin{pmatrix}\cos(\omega t/2) & -\sin(\omega t/2)\\ \sin(\omega t/2) & \cos(\omega t/2)\end{pmatrix}\mathbf{R}. \tag{21}$$

This soliton is located at $2\mathbf{R}(t)$ and makes a circular motion with the radius R and the velocity V, with

$$V = \omega R. \tag{22}$$

It is also *breathing*, which means an oscillation of the scale of the soliton; the scale parameter r_0 is the amplitude of the breathing mode.

The frequency of the cyclotron motion is $eQB/M = eB/m = \omega$, as is expected. The only unexpected result is the breathing mode. However, the breathing is obvious from a mathematical point of view, since our time-dependent solutions are constructed by performing a time-dependent dilation (5) to the static solutions. Therefore, any soliton solutions in the system (10) may begin to breath when the external magnetic field B is switched on. This is true even for the model with a general value of the coupling constant, $g \neq g_c$, although no analytic solutions are known in such a model. Furthermore, this conclusion is applicable to any dilation-invariant theory.

The energies of the above time-dependent solitons are calculated as

$$E = E_S + E_B + E_C, \tag{23}$$

where

$$E_S = -\frac{eB}{m}\frac{\pi}{\alpha}N, \tag{24}$$

$$E_B = \frac{e^2B^2\pi^2r_0^2}{4|\alpha|m\sin(\pi/N)}, \tag{25}$$

$$E_C = MV^2/2, \tag{26}$$

with (18) and (22). The third term E_C is the kinetic energy of the cyclotron motion, which confirms the identification of (18) as the mass of the soliton. The second term E_B is understood as the energy of the breathing mode because it is proportional to the square of the amplitude r_0 of the breathing mode; note that it vanishes as $r_0 \to 0$. The first term E_S is independent of r_0 and V, and therefore is independent of the breathing mode and the cyclotron motion. It is reasonable to understand E_S as the energy of a spin–magnetic-field interaction. Then, we may rewrite this term as

$$E_S = -g_S(eB/2m)S, \tag{27}$$

where S is the spin of the soliton, (17), and $g_S = 1$. It follows that the g factor of the soliton is 1.

We also calculate the current $\mathbf{I}$ of the soliton solution:

$$\mathbf{I} = (1/2m)i[\psi^\dagger\mathbf{D}\psi - (\mathbf{D}\psi)^\dagger\psi], \tag{28}$$

with $D_i = \partial_i - ia_i + ieA_i$. A particularly interesting quantity is the θ component of $\mathbf{I}$. It is an internal current associated with its spinning. Hence, we set $\mathbf{R} = 0$ in (21). Then, the soliton is breathing but its center of mass is at rest, for which we find that

$$\int_0^\infty dr\, I_\theta = -\frac{NeB}{2m|\alpha|} = -\frac{\omega}{4\pi}Q, \tag{29}$$

where Q is the statistical charge of the soliton. The result might be interpreted as the current generated by a spinning disk with the frequency $\omega/2$ where the charge Q is distributed. However, this frequency is half of the proper frequency of the soliton.

Another important quantity is the orbital angular momentum of our rotating solitons. To calculate this quantity we shift the center of the cyclotron motion to the origin of the coordinate. This induces a phase of the solution such that

$$\psi \to \exp[-i(eB/2)\varepsilon_{ij}x_iR_j]\psi. \tag{30}$$

Substituting this into the total angular momentum (15), we obtain

$$J = S + eB\pi R^2N/|\alpha| = S + MVR/2, \tag{31}$$

where S, M, and V are the spin, the mass, and the velocity of the soliton: R is the radius of the cyclotron motion. It is curious that the orbital angular momentum is half of what is naively expected, i.e., MVR. As we now show, precisely this result leads to a consistent quantization of the orbital angular momentum. The result (31), together with (29), implies that the motion of the soliton cannot be simply simulated by a spinning particle making a cyclotron motion.

We finally consider a Bohr-Sommerfeld semiclassical quantization; see, e.g., Ref. [14]. It is trivial to see that the action of our time-dependent solution vanishes. This is because the value of the action is not modified by switching on the external magnetic field; see (1) and (9). Consequently, we achieve the following quantization of each mode of the breathing and cyclotron motion:

$$E_BT = 2\pi p, \quad E_CT = 2\pi q, \tag{32}$$

with integers p and q; here, $T = 2\pi/\omega$ is the period of the motion. It turns out that quantum solitons are in the Landau levels. By these quantization conditions, the free parameters r_0 and R of the breather soliton are quantized and become functions of m and B. Condition (32) also implies that the orbital angular momentum becomes integer, as it should:

$$MVR/2=q\,. \tag{33}$$

Therefore, the semiclassical quantization of the cyclotron motion leads to a consistent quantization of the angular momentum.

In this paper we analyzed the CS vortex solitons in the external magnetic field B. We have shown that, if the system without B is invariant under dilation, it is possible to construct soliton solutions in the system with B from those in the system without B by performing the transformation (8). The resulting solution is time dependent, which describes a breather soliton making a cyclotron motion. In particular, we have obtained analytic time-dependent solutions in the JP model by switching on the external magnetic field. Semiclassical quantization of these time-dependent solitons has also been made successfully. Although we have only considered single-soliton solutions in this paper, it is trivial to generalize our method to construct time-dependent multisoliton solutions in the external magnetic field simply by using the JP multisoliton solutions [4].

We would like to thank S. Takagi for helpful discussions on the subject of this paper.

Note added.—After submitting this Letter we learned from Jackiw and Pi that the time-dependent dilation (5) was previously considered by Jackiw [15]. They also informed us that using a different gauge from ours they have applied a semiclassical quantization to our time-dependent periodic solitons [16]. Their result does not coincide with ours. As they point out this is due to the gauge noninvariance of the CS action. This noninvariance comes from nonvanishing boundary contributions in the time direction associated with partial integrations, while boundary contributions in the space direction should vanish when the gauge invariance of the Lagrangian $L=\int d^2x\,\mathcal{L}$ is imposed at the classical level. Let us consider a gauge transformation, $\psi\to e^{i\Lambda}\psi$, $a_\mu\to a_\mu+\partial_\mu\Lambda$. In applying semiclassical quantization, the allowed gauge transformations are such that $\Lambda(t+T)=\Lambda(t)+2\pi l$ with l being an integer; here, T is the period of the classical motion. Now, it is trivial to see that the action is transformed as $\int d^3x\,\mathcal{L}\to\int d^3x\,\mathcal{L}-\pi lQ$, where the time integration is over one period. When a gauge transformation is performed, we obtain classically equivalent solutions. However, quantum mechanically, they are different objects if the action is different. We are led to conclude that in the semiclassical treatment the U(1) gauge transformations in the soliton sector are classified into infinitely many inequivalent classes indexed by an integer l. Corresponding to each class there is a quantum soliton whose action is given by πlQ. We need to take into account all of these solitons. It is easy to see that the quantization condition $E_CT=2\pi q$ for the cyclotron mode holds even if the action is nonzero. Then, the total energy of the soliton reads $E=\omega(p+q)-(\alpha l/|\alpha|)(eB/2m)S$. It implies that the g factor of the quantum soliton may be an arbitrary integer l.

We acknowledge correspondence with R. Jackiw and S.-Y. Pi which motivated us to consider the problem of inequivalent gauges in the CS soliton system, although their formalism is different from ours on this point.

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