Ward Identities and the β Function in the Luttinger Liquid

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One-dimensional metals have a particular symmetry associated with the discrete structure of the Fermi surface: separate charge conservation in low-energy-scattering processes for particles near the left and right Fermi points, respectively. The field-theoretic renormalization group allows for an efficient exploitation of the Ward identities following from this symmetry. As a first application we prove that the β function of the Luttinger model vanishes identically. The same symmetry ensures the finiteness of the compressibility, thus making possible the existence of stable metallic phases with anomalous dimensions in $d=1$.

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The unusual properties of high-T_c cuprates in the nonsuperconducting phase have led to the recent proposal of a possible breakdown of Fermi-liquid theory in two dimensions [1,2], thus stimulating a reinvestigation of the mechanisms governing the non-Fermi-liquid phases known in $d=1$.

The breakdown of Fermi-liquid theory in one-dimensional interacting Fermi systems shows up already at second-order perturbation theory: The corrections to the vertex functions diverge logarithmically near the Fermi surface of the noninteracting system. A particular renormalization-group approach, known as "g-ology," was developed in the 1970s [3], where the important interactions are parametrized by a small set of coupling constants g_i. The divergences are then handled by scaling towards exactly soluble models such as the Luttinger model [4]. A scaling ansatz has been assumed for the vertex functions, allowing one to approach the Fermi surface by rescaling the fields and coupling constants. The validity of the ansatz has been verified to that order in the couplings to which explicit calculations have been carried out [3].

For spinless fermions in a one-dimensional continuum it has been rigorously shown that the β function is analytic in the coupling constant [5]. Even more, the β function is known to be zero to three-loop order in this case [3]. Clearly, the identical vanishing of the β function to *all* orders can be easily deduced by resorting to the exact solution [6] for the propagator of the Luttinger model, which describes the low-energy physics of generic spinless one-dimensional continuum fermions: The exact exponent η describing the low energy-momentum asymptotics of the propagator is a continuous function of the coupling constant, which is possible only if the β function vanishes identically, thus giving rise to a line of fixed points.

The purpose of this Letter is threefold. First, we rewrite the old scaling approach [3] to one-dimensional Fermi systems in the language of the field-theoretic renormalization group familiar from critical phenomena [7,8]. Renormalizability is easily shown in the latter formulation, and implies that the scaling ansatz used in g-ology indeed holds to *all* orders in the couplings. Second, we show how Ward identities can be used to obtain all-order constraints on the structure of the renormalization group. In particular, we prove that the β function for spinless fermions vanishes identically. This proof, which makes no use of the exact solution of the Luttinger model, has the merit of identifying the two salient ingredients leading to a line of fixed points with non-Fermi-liquid behavior, namely, (i) divergent terms in perturbation theory and (ii) a symmetry of the interaction term (charge conservation, separately in each channel) yielding the cancellation of divergences in the renormalized coupling constant. Finally, we point out that the separate charge conservation plays another important role: It ensures the finiteness of the compressibility, thus making possible the existence of stable one-dimensional metallic phases —despite the presence of a propagator with anomalous dimensions.

The low-energy physics of spinless fermions in one dimension (extensions will follow in a longer publication) is described by the Luttinger model [4]

$$H=H_0=H_I\,, \tag{1a}$$

$$H_0=\sum_{\alpha=\pm}\sum_{\mathbf{k}}\epsilon_\alpha(\mathbf{k})a^\dagger_\alpha(\mathbf{k})a_\alpha(\mathbf{k})\,, \tag{1b}$$

$$H_I=\frac{g}{V}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}}a^\dagger_+(\mathbf{k}+\mathbf{q})a_+(\mathbf{k})a^\dagger_-(\mathbf{k}'-\mathbf{q})a_-(\mathbf{k}')\,. \tag{1c}$$

Here $a^\dagger_\pm(\mathbf{k})$ $[a_\pm(\mathbf{k})]$ are creation (annihilation) operators for fermions near $\pm k_F$ ("channel" $\alpha=\pm$), where $\mathbf{k}$ is measured relative to $\pm k_F$; $\epsilon_\pm(\mathbf{k})=\pm v_F\mathbf{k}$ is a linearized dispersion (v_F Fermi velocity), g is a coupling constant, and V is the volume of the system. We keep the vector notation (boldface) for momenta even in one dimension to distinguish them from the bivectors $k=(\mathbf{k},\omega)$ which include the energy variable. Nonlinear corrections to $\epsilon_\pm(\mathbf{k})$ and $\mathbf{k}$-dependent interactions scale to zero at low energies, i.e., they are irrelevant in the renormalization-group sense [3]. Because of the Pauli principle there is no interaction term involving fermions near $+k_F$ (or $-k_F$) only.

The model described by Hamiltonian (1) is well defined only if an ultraviolet cutoff is introduced. We impose a bandwidth cutoff Λ by requiring $|\mathbf{k}|<\Lambda$. Note that the scale k_F does not appear in the Luttinger model. Only the *existence* of two Fermi points is important, not their *distance* in momentum space. Hence, Λ^{-1} *is the only length scale in the system.* The disappearance of the length scale k_F^{-1} is here obtained free; in $d>1$, where the Fermi surface is a continuum, its elimination is more involved [5].

Our investigation involves the following Green functions, defined as ground-state expectation values of time-ordered operator products:

$$G_\alpha(x,x') = G_\alpha^{(2)}(x,x') = -i\langle \mathcal{T}\Psi_\alpha(x)\Psi_\alpha^\dagger(x')\rangle\,, \tag{2a}$$

$$G_{\alpha\beta}^{(4)}(x_1,x_2;x_1',x_2') = \langle \mathcal{T}\Psi_\alpha(x_1)\Psi_\beta(x_2)\Psi_\alpha^\dagger(x_1')\Psi_\beta^\dagger(x_2')\rangle\,, \tag{2b}$$

$$G_{\alpha,\gamma}^{(2,1)}(x_1,x_1';y) = -\langle \mathcal{T}\Psi_\alpha(x_1)\Psi_\alpha^\dagger(x_1')\rho_\gamma(y)\rangle\,, \tag{2c}$$

and analogously for $G^{(4,1)}$ and $G^{(0,2)}$. Here $\Psi_\alpha^\dagger(x)$ $[\Psi_\alpha(x)]$ are Heisenberg operators at the space-time point $x=(\mathbf{r},t)$, where the corresponding Schrödinger operators $\Psi_\alpha^\dagger(\mathbf{r})$ $[\Psi_\alpha(\mathbf{r})]$ are the Fourier transforms of $a_\alpha^\dagger(\mathbf{k})$ $[a_\alpha(\mathbf{k})]$; $\rho_\alpha(x)=\Psi_\alpha^\dagger(x)\Psi_\alpha(x)$ is the charge-density operator. The vertex functions $\Gamma^{(n,l)}$ corresponding to $G^{(n,l)}$ are constructed by considering only connected, one-particle-irreducible Feynman diagrams with amputated external legs. The respective Fourier components are denoted by $\Gamma^{(2)}(p)$, $\Gamma^{(4)}(p_1,p_2;p_1',p_2')$, $\Gamma^{(2,1)}(p,p';q)$, etc. For $g=0$, $\Gamma^{(2)}$ is given by $\Gamma_{0\pm}^{(2)}(p)=\not{p}_\pm$, where $\not{p}_\pm=\omega\mp v_F\mathbf{p}$.

Perturbative results for the vertex functions diverge at low energy-momenta, i.e., a renormalization-group treatment is necessary [3].

Since Λ^{-1} is the only length scale in the system, dimensionless quantities depend only via p/Λ on p and Λ. Hence, *the low energy-momentum limit $p\to 0$ is directly related to the ultraviolet limit $\Lambda\to\infty$.* The ultraviolet (UV) point of view is more convenient since the UV degree of divergence of diagrams can be determined by simple power counting.

The canonical dimensions associated with the Hamiltonian (1) and the fields are $[H]=\Lambda^1$ and $[a_\alpha(\mathbf{k})]=\Lambda^{-1/2}$, respectively, and therefore $[g]=\Lambda^0$, i.e., the coupling g is dimensionless and the bare propagator $G_{0\alpha}(p)$ has dimension Λ^{-1}. We define $\gamma^{(2)}=\Gamma^{(2)}/\not{p}$ and $\gamma^{(4)}=\Gamma^{(4)}/g$, which are both dimensionless and normalized to 1 in the noninteracting limit. Power counting predicts that primitive divergences are possible in $\Gamma^{(2)}$, $\Gamma^{(0,2)}$, $\Gamma^{(2,1)}$, and $\Gamma^{(4)}$, but not in higher-order vertex functions. An explicit calculation shows, however, that $\Gamma^{(0,2)}$ is actually primitively convergent (there is only one skeleton diagram, the particle-hole bubble, which is finite for $\Lambda\to\infty$). Hence, at most three renormalization factors are necessary to remove the UV divergences from the theory. We define renormalized vertex functions [7]

$$\bar\gamma^{(i)}(p/\lambda,\bar g) = \lim_{\Lambda\to\infty} Z^{(i)}(\lambda/\Lambda,g)\gamma^{(i)}(p/\Lambda,g)\,, \tag{3}$$

where $(i)=(2),(4)$, respectively, and $\bar g$, the renormalized coupling constant, is given by

$$\bar g = [(Z^{(2)})^2/Z^{(4)}]g\,. \tag{4}$$

The renormalization factors $Z^{(i)}$ are chosen such that $Z^{(i)}\gamma^{(i)}$ is finite at any finite order in $\bar g$ (as $\Lambda\to\infty$); p in (3) represents all external bimomenta of the vertex functions and λ parametrizes the renormalization-group transformation [7] (see below). The property that all UV divergences can indeed be absorbed in a finite number of p-independent renormalization factors is a nontrivial feature of a model, its *renormalizability*. In general, a model is renormalizable if the coupling constants have non-negative canonical scaling dimensions [8], which is the case for the problem at hand, since g is dimensionless.

The requirement of finiteness does not determine the renormalization factors $Z^{(i)}$ uniquely. To fix the choice one can impose "normalization conditions" on the renormalized vertices $\bar\gamma^{(i)}$ by requiring that $\bar\gamma^{(i)}(\mathrm{NP},\bar g)=1$ at a special point in p space, the "normalization point" (NP). Different normalization points yield different renormalized theories which all correspond to the same bare theory. The transformations from one renormalized theory to another make up the field-theoretic renormalization group [7,8]. To obtain scaling equations relating high and low energy-momentum regimes it is sufficient to consider a one-parameter family of normalization points, which can be parametrized by a single parameter λ. Moving from λ to λ', the renormalized vertex functions transform as

$$\bar\gamma^{(i)}(p/\lambda',\bar g') = z^{(i)}(\lambda'/\lambda,\bar g)\bar\gamma^{(i)}(p/\lambda,\bar g)\,, \tag{5}$$

where $z^{(i)}=\lim_{\Lambda\to\infty}[Z^{(i)}(\lambda'/\Lambda,g)/Z^{(i)}(\lambda/\Lambda,g)]$ which can be expressed as a function of λ'/λ and $\bar g$. The renormalized coupling transforms as

$$\bar g' = [(z^{(2)})^2/z^{(4)}]\bar g\,. \tag{6}$$

Equations (5) and (6) correspond to the old *scaling ansatz* [3], which therefore holds to all orders in the coupling constant. The infinitesimal variation of the renormalized coupling defines the β function,

$$\beta(\bar g) = \frac{\partial}{\partial(\lambda'/\lambda)}\left(\frac{(z^{(2)})^2}{z^{(4)}}\bar g\right)_{\lambda'=\lambda}. \tag{7}$$

The scaling dimension x_Ψ of the field $\Psi_\alpha(x)$ is given by the bare dimension $x_\Psi^0=\frac{1}{2}$ plus the anomalous part $\eta/2$, which is obtained by differentiating (5) for $\bar\gamma^{(2)}$, as [7]

$$\frac{\eta}{2} = -\left(\frac{\partial[(z^{(2)})^{-1/2}]}{\partial(\lambda'/\lambda)}\right)_{\lambda'=\lambda,\bar g=\bar g^*}, \tag{8}$$

where $\bar g^*$ is the fixed-point solution of $\beta(\bar g^*)=0$ attracting the starting coupling $\bar g$ upon scaling towards the Fermi surface. The scaling dimension determines the large distance (i.e., small energy-momentum) asymptotics of the propagator G. A positive anomalous dimension implies non-Fermi-liquid behavior of the system, where both the quasiparticle weight and the single-particle density of states vanish at ω^{η} near the Fermi surface, while the momentum distribution loses its jump in favor of a $\mathbf{k}^{\eta}$-power-law behavior.

We will now use Ward identities to show that the β function (7) of the Luttinger model vanishes identically to all orders in the coupling constant.

The irreducible charge vertex $\Lambda^{(2,1)}$, i.e., the sum of all one-interaction-irreducible contributions to $\Gamma^{(2,1)}$, obeys the Ward identities [9,10]

$$q\!\!\!/_{\alpha}\Lambda^{(2,1)}_{\alpha,\alpha}(p,p+q;q)=\Gamma^{(2)}_{\alpha}(p+q)-\Gamma^{(2)}_{\alpha}(p)\,, \tag{9a}$$

$$\Lambda^{(2,1)}_{\alpha,-\alpha}(p,p+q;q)=0\,. \tag{9b}$$

These identities follow from the separate charge conservation in each channel α in momentum space and can be proved via the equation of motion for $G^{(2,1)}$ [9]. Using (9) one can show that $D_{\alpha\beta}(q)$, the effective interaction between the channels α and β, can be calculated exactly by performing the RPA sum with the *bare* polarization bubble $\Pi^{0}_{\pm}(q)$, since due to (9) all vertex and self-energy corrections to Π^0 cancel each other [10]. In particular, D remains finite as $\Lambda\to\infty$, i.e., D need not be renormalized. Denoting the renormalization factor of $\Lambda^{(2,1)}$ by $Z^{(2,1)}$, the Ward identity (9a) implies

$$Z^{(2,1)}=Z^{(2)} \tag{10}$$

up to a finite, scale-invariant constant. We note that (10) holds in any interacting Fermi system as a consequence of the usual charge conservation, which leads to a Ward identity relating the charge and current vertices to $\Gamma^{(2)}$. However, the validity of the two separate identities (9) and the ensuing finiteness of D are a consequence of the separate charge conservation specific to the Luttinger model.

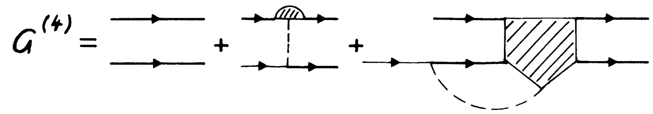

FIG. 1. Dyson equation for $G^{(4)}$; bold (thin) lines represent G (G_0); dashed lines, g; the shaded semicircle, $\Gamma^{(2,1)}$; and the shaded pentagon, $\Gamma^{(4,1)}$.

To relate $Z^{(4)}$ to $Z^{(2,1)}$ and $Z^{(2)}$ we consider the skeleton structure of $\Gamma^{(4)}$ in terms of dressed Green functions G, dressed interactions D, and dressed charge vertices $\Lambda^{(2,1)}$. We now show that $\Gamma^{(4)}$ has no primitive divergences, i.e., all divergences contributing to $\Gamma^{(4)}$ are due to G and $\Lambda^{(2,1)}$ insertions.

Naive power counting predicts logarithmic UV divergences for the $\Gamma^{(4)}$ skeletons. However, when all diagrams of a given order are summed, these divergences cancel each other. To show this, we rewrite $G^{(4)}$ by combining the Dyson equation shown in Fig. 1 (derived from the equation of motion for $G^{(4)}$) with the following Ward identity for $\Gamma^{(4,1)}$:

$$g\Gamma^{(4,1)}_{\alpha_1\alpha_2,\beta}(p_1,p_2;p_1',p_2';q)=\frac{D_{-\beta,\alpha_1}(q)}{q\!\!\!/_{\alpha_1}}\left\{\frac{G_{\alpha_1}(p_1+q)}{G_{\alpha_1}(p_1)}\Gamma^{(4)}_{\alpha_1\alpha_2}(p_1+q,p_2;p_1',p_2')-\frac{G_{\alpha_1}(p_1'-q)}{G_{\alpha_1}(p_1')}\Gamma^{(4)}_{\alpha_1\alpha_2}(p_1,p_2;p_1'-q,p_2')\right\}+(1\leftrightarrow 2)\,. \tag{11}$$

This identity is also [as (9)] a consequence of separate charge conservation in each channel and can be derived from the equation of motion for $G^{(4,1)}$. Inserting (11), the last term in Fig. 1 can be expressed in terms of $\Gamma^{(4)}$ as shown in Fig. 2; a wiggly line represents $D_{\alpha\beta}(q)/q\!\!\!/_{\beta}$. The first term on the right-hand side in Fig. 2 vanishes, since $\int_q D_{\alpha\alpha}(q)/q\!\!\!/_{\alpha}=0$. Figures 1 and 2 provide an exact integral equation for the four-point vertex of the Luttinger model. Performing power counting for the skeleton diagrams contributing to the remaining three terms in Fig. 2, one realizes that the degree of divergence has been reduced by one, since the wiggly lines vanish as Λ^{-1} for large momenta (in contrast to D, which behaves as Λ^0). Hence $\Gamma^{(4)}$ has no primitive divergences, i.e., it can be renormalized by just renormalizing the $\Lambda^{(2,1)}$ and G insertions entering its skeleton structure. This amounts to multiplying each $\Lambda^{(2,1)}$ by $Z^{(2,1)}$ and each G by $(Z^{(2)})^{-1}$ [since $G=(\Gamma^{(2)})^{-1}$]. Since an nth-order $\Gamma^{(4)}$ skeleton contains $2n$ $\Lambda^{(2,1)}$ vertices and $2n-2$ G lines and since $Z^{(2,1)}=Z^{(2)}$, this corresponds to an overall multiplication by $(Z^{(2)})^2$—independent of n. Thus

$$Z^{(4)}=(Z^{(2)})^2 \tag{12}$$

up to a finite, scale-independent factor. This yields, by (4), $\bar{g}=g$ and therefore

$$\beta(\bar{g})\equiv 0\,. \tag{13}$$

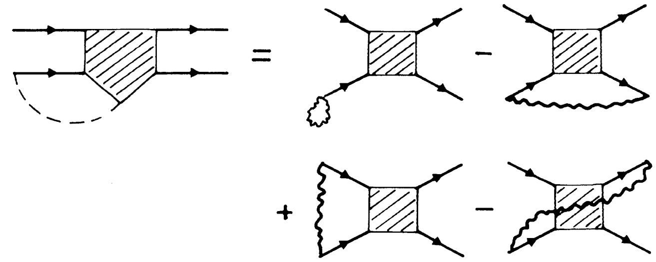

FIG. 2. Decomposition of the last term in Fig. 1; the shaded rectangles represent $\Gamma^{(4)}$; the wiggly lines, $D(q)/q\!\!\!/$.

This means that g remains marginal, i.e., the Luttinger model is governed by a line of fixed points. We note that for a general choice of normalization conditions $\bar{g}$ becomes a finite, scale-independent function of g, but the result $\beta \equiv 0$ is independent of this choice.

There are instructive analogies relating the present theory to quantum electrodynamics (QED). In QED the coupling (the fine-structure constant α) is also dimensionless and primitive divergences are present in the electron propagator G, the photon propagator D, and the electron-photon vertex $\Lambda^{(2,1)}$, while the four-electron vertex is primitively convergent. Local charge conservation leads to a Ward identity yielding a cancellation of the G and $\Lambda^{(2,1)}$ renormalizations in the renormalization of α. As a consequence, the coupling constant in QED renormalizes as the photon propagator D. The crucial difference in the Luttinger model is that D is finite, i.e., the coupling does not renormalize at all. A similar complete cancellation of coupling-constant renormalizations due to Ward identities is known to occur in the Thirring model [11].

Of course, Ward identities can also be exploited for more general models of interacting fermions, where exact solutions are not available. In particular, in a one-dimensional model with spin and backscattering but without umklapp scattering, separate charge conservation still holds in each channel and yields Ward identities for the vertex functions, thus again reducing the number of independent renormalization factors. This will be the subject of a longer paper.

Many physical quantities in the Luttinger model show a peculiar critical behavior [3]. The quasiparticle weight and the single-particle density of states both vanish as a power law near the Fermi surface, the density-density response function near $q = 2k_F$ diverges (vanishes) for positive (negative) g, and the Cooper pair correlation diverges (vanishes) for negative (positive) g. Nevertheless, the *compressibility*, i.e., the static, homogeneous limit of the density-density response remains *finite*, as expected on physical grounds. This is not trivial, since there are singular terms in the perturbative structure of the density-density response. Total charge conservation leads to a Ward identity which guarantees the cancellation of singular contributions order by order in perturbation theory. The Ward identities following from the separate charge conservation of left and right particles ensures also the absence of singular behavior of nonperturbative origin. In fact, the exact density-density response of the Luttinger model is given by an RPA resummation of bubbles with bare Green functions [3], i.e., the anomalous dimension of the full propagator is compensated exactly by the vertex corrections. This latter result can be extended to fermions with spin, as long as umklapp processes which violate the separate charge conservation are absent.

Recently, a renormalization group for d-dimensional fermions has been constructed by following Wilson's procedure, i.e., integrating out high momenta [5,12]. However, the only singularity found so far in $d > 1$ is the well-known Cooper instability, which appears at one-loop order in $\Gamma^{(4)}$. Singularities or anomalous behavior of the propagator G leading to a breakdown of Fermi-liquid theory have not yet been explicitly identified. If there are, there must be some mechanism which cancels their effect on the density-density response function to prevent an anomalous behavior of the compressibility. The symmetry leading to such a cancellation in the Luttinger model is specific to the discrete structure of the Fermi surface in one dimension and can hardly be generalized to higher dimensions, thus making the possibility of anomalous behavior of the propagator rather remote.

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[1] C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, Phys. Rev. Lett. **63**, 1996 (1989).

[2] P. W. Anderson, Phys. Rev. Lett. **64**, 1839 (1990); **65**, 2306 (1990).

[3] For a review, see J. Solyom, Adv. Phys. **28**, 201 (1979).

[4] S. Tomonaga, Prog. Theor. Phys. **5**, 544 (1950); J. M. Luttinger, J. Math. Phys. **4**, 1154 (1963).

[5] G. Benfatto and G. Gallavotti, Phys. Rev. B **42**, 9967 (1990); J. Stat. Phys. **59**, 541 (1990).

[6] A. Theumann, J. Math. Phys. **8**, 2460 (1967).

[7] For a review, see C. Di Castro and G. Jona-Lasinio, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1976), Vol. 6.

[8] For an introduction to the field-theoretic renormalization group, see D. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (World Scientific, Singapore, 1984).

[9] H. U. Everts and H. Schulz, Solid State Commun. **15**, 1413 (1974).

[10] I. E. Dzyaloshinskii and A. I. Larkin, Zh. Eksp. Teor. Fiz. **65**, 411 (1973) [Sov. Phys. JETP **38**, 202 (1974)].

[11] M. Gomes and J. H. Lowenstein, Nucl. Phys. **B45**, 252 (1972).

[12] For a pedagogical description to one-loop order, see R. Shankar, Physica (Amsterdam) A (to be published).