Experimental Determination of a Nonlinear Hamiltonian in a Synchrotron

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The nonlinear beam dynamics of transverse betatron oscillations were studied experimentally at the Indiana University Cyclotron Facility Cooler Ring. Particles were kicked onto resonance islands and the properties of these islands were studied. The island tune was determined with high precision by Fourier analyzing the spectrum containing the island oscillations. The island width was estimated based on a single-resonance model. The Hamiltonian of particle motion near a resonance condition was thus deduced.

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In recent years, nonlinear mechanics has been studied in various subfields of physics. Nonlinear-beam-dynamics studies have been especially important in the design of future colliders such as the Superconducting Super Collider (SSC) and the Relativistic Heavy Ion Collider (RHIC), since the higher-order multipoles in superconducting magnets are considerably greater than those in conventional iron magnets. Theoretical studies [1] of nonlinear fields have been used to predict both the long- and the short-term behavior of orbiting particles in an accelerator. In order to better understand the approximations used in theoretical predictions, experimental studies of resonant behavior are essential.

Several nonlinear-beam-dynamics experiments have been performed in the past [2]. These experiments studied general features of nonlinear motion, such as resonance island stability, smear, tune dependence on amplitude, etc. However, the island structure has not yet been studied in detail. This Letter reports some recent results of nonlinear-beam-dynamics experiments performed at the Indiana University Cyclotron Facility (IUCF) Cooler Ring. We present a method for obtaining properties of resonance islands and derive the Hamiltonian for the particle motion near a resonance condition.

For particle motion in a circular accelerator, the horizontal deviation from the closed orbit, $x(s)$, satisfies Hill's equation:

$$\frac{d^2x}{ds^2}+K(s)x=\frac{\Delta B_y}{B\rho}\,. \tag{1}$$

Here $K(s)$ is the quadrupole strength, $B\rho=p/e$ is the momentum rigidity, and s is the longitudinal particle coordinate, which advances from 0 to C ($=2\pi R$), the circumference, as the particle completes one revolution of the cyclic accelerator, where R is the average radius. The higher-order anharmonic term, $\Delta B_y/B\rho$, which arises from higher-order multipoles, is normally small. Oscillations about the closed orbit due to the linear focusing force of quadrupoles, $K(s)$, are called betatron oscillations. The number of oscillation periods in one revolution is the betatron tune, ν_x, which can be adjusted by varying the quadrupole strength of the accelerator. Both $K(s)$ and the anharmonic term $\Delta B_y/B\rho$ are periodic functions of s with period C. In proton accelerators, the damping of the phase-space motion due to synchrotron radiation is negligible; hence the phase-space area of particle motion is conserved.

Neglecting the small anharmonic term in the Hamiltonian, the betatron motion is linear. Hill's equation (1) can be solved [3] using the Floquet transformation to obtain the solution $x=(2\beta_x J)^{1/2}\cos\phi$, where J and ϕ are action-angle variables. Here $2J$ is the phase-space area (called the Courant-Snyder invariant or the emittance) of the betatron motion and β_x is the betatron amplitude function of the Floquet transformation (β_x is periodic in s with period C). For each turn around the accelerator, the angular variable ϕ increases by $2\pi\nu_x$, where ν_x is the horizontal betatron tune. The conjugate phase-space coordinate, $x'=dx/ds$, can be determined using two beam-position monitors (BPMs). The turn-by-turn tracking of motion in (x,x') phase space as observed at a given location in the cyclic accelerator is called the Poincaré map. Betatron oscillations resulting from a linear force produce ellipses in the Poincaré map.

Nonlinear perturbations in the accelerator include sextupole fields in dipoles, chromaticity correction sextupoles, octupoles, and some small higher-order random error multipoles. These anharmonic terms usually do not significantly perturb the particle motion in phase space except when the betatron tunes are near to a resonance condition, which occurs at $m\nu_x = n$ for one-dimensional motion, where m, n are integers. The Poincaré map deviates from an ellipse at a resonance condition, where stable particle motion around fixed points (a stable solution to the equation of motion) in phase space bounded by invariant surfaces may occur for nearly integrable Hamiltonian systems. These stable phase-space ellipses (called islands) around fixed points are separated by the unstable fixed points. The particle phase-space trajectory passing through unstable fixed points is called the separatrix.

The IUCF Cooler Ring provides an ideal environment for nonlinear-beam-dynamics experiments. The Cooler Ring is hexagonal with a circumference of 86.82 m. The relative momentum spread of the beam is about ± 0.0001. The 95% emittance, or phase-space area, of the proton beam is electron cooled to much less than 1 π mm mrad in about 3 s. The beam lifetime can be as long as hours.

The experimental procedure started with a single bunch of about 3×10^8 protons with a kinetic energy of 45 MeV. The cycle time was 10 s. The injected beam was electron cooled for about 3 s. The bunch length was about 3.6 m (or 40 ns) and its period of revolution was 969 ns. The rf frequency was 1.031 68 MHz. The beam was kicked with various angular deflections, θ_K, by a pulsed deflecting magnet with pulse width of 500 ns and a rise and fall time of 100 ns. The electron-cooling system was turned off 20 ms before kicking. The subsequent beam-centroid displacement was measured by two BPMs, with an rms position resolution of about 0.1 mm. The stability of the horizontal closed orbit was measured to be less than 0.02 mm. The turn-by-turn beam positions were digitized and recorded in transient recorders. A total of 4096 turns was recorded in the available memory buffer for each kick.

The resonance structure was investigated using different orbit deflector strengths. Transverse displacements $(x_1, x_2)_n$ were measured at the nth turn in the two BPMs. The relative betatron amplitude functions and the betatron phase advance between the two BPMs were deduced from the turn-by-turn data of (x_1, x_2). The phase-space coordinates were then transformed to the normal coordinates $(x_1, p_{x1})_n$, where $p_{x1} = -\frac{1}{2}(d\beta_x/ds)x_1 + \beta_x x_1'$. For linear betatron motion, the phase-space ellipse in the normal coordinates is a circle [3]. Figure 1 shows the Poincaré map in the normal coordinates, where the betatron tune is $\nu_x = 3.7578$ for the left graph and $\nu_x = 3.7500$ for the right graph. The Poincaré map in the right part of the figure shows that particles were kicked onto the fourth-order resonance islands. The particle motion returned to the same island every fourth turn. Within an island the particle trajectory traced out an ellipse around the corresponding stable fixed point. Because of the inherent coupling of horizontal and vertical betatron motion, the ellipse around the stable fixed point in an is-

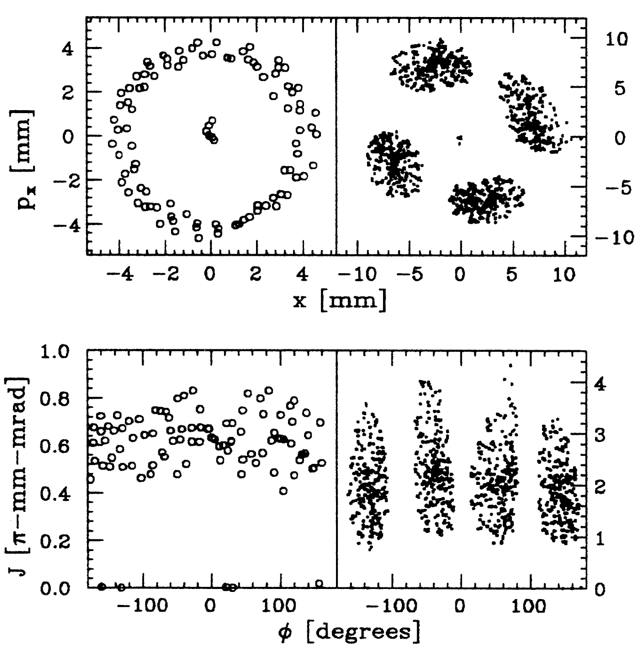

FIG. 1. The Poincaré maps in the normal coordinates, (x_1, p_{x_1}), at the betatron tunes $\nu_x = 3.7578$ (left) and $\nu_x = 3.7500$ (right) are shown for comparison. The resolution of the measurement is about 0.1 mm. The corresponding maps using the action-angle variables (J_1, ϕ_1) are also shown in the lower part of the figure.

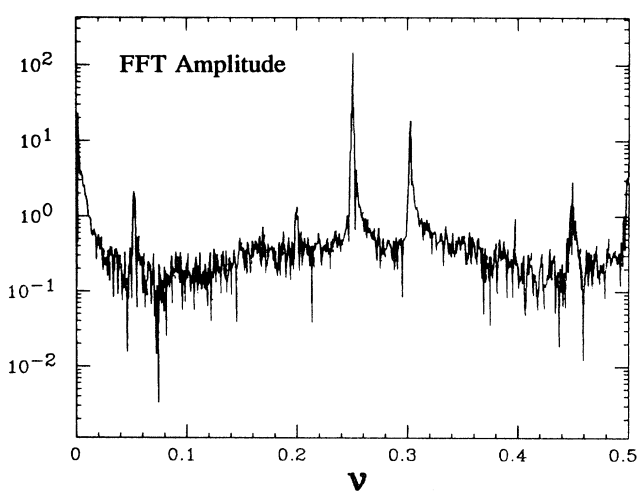

FIG. 2. The FFT spectrum for the betatron motion at the resonance condition $4\nu_x = 15$. The FFT spectrum shows the fractional part of the betatron tunes. Because of the linear coupling, a vertical betatron tune peak is also seen. Along with the betatron tune lines, the FFT spectrum shows many interesting smaller peaks, which are due to oscillations around the island fixed points.

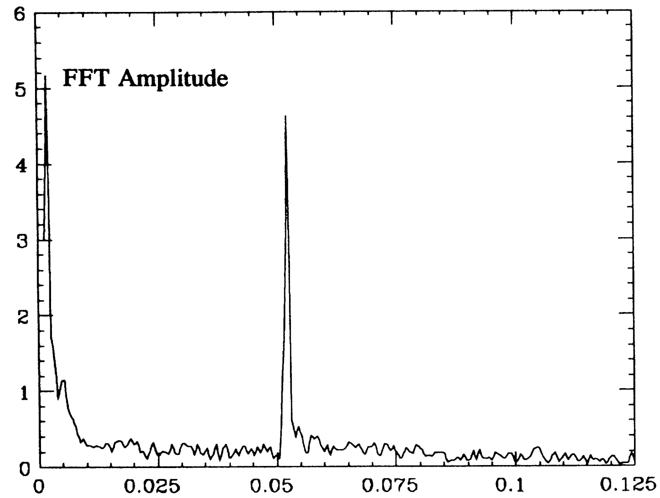

FIG. 3. The FFT spectrum of the motion around a fixed point in an island. The island tune is $\nu_{\rm island}=0.0013$.

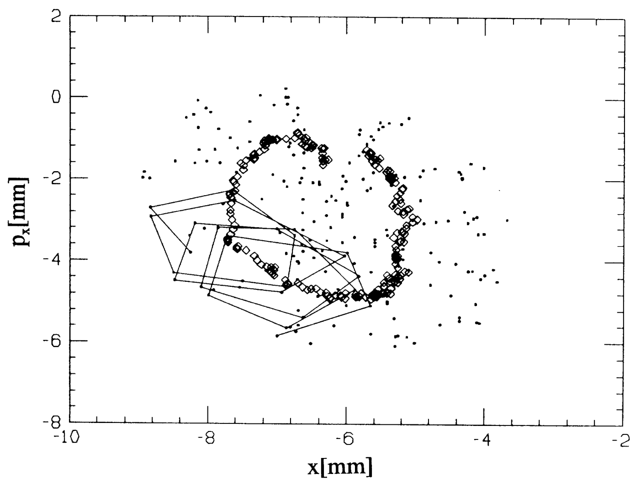

FIG. 4. The phase-space points (dots) of the island in the third quadrant shown on the right-hand side of Fig. 1 are displayed with the corresponding five-island-turn running average (diamonds). The averaged five-island-turn centroids move along an ellipse around a stable fixed point of the fourth-order resonance.

land is smeared.

The fast-Fourier-transform (FFT) spectrum of the betatron motion at the fourth-order resonance, $\nu_x=3.7500$, is shown in Fig. 2. Note that the vertical betatron tune due to linear betatron coupling is also observed at $\nu_y=5-0.3124$. The ratio of the betatron peaks in the Fourier spectrum is a measure of the amount of betatron coupling. We purposely moved the vertical betatron tune away from the horizontal tune to reduce the effect of the betatron coupling.

The frequency of oscillation about the island provides useful information. Using the data of Fig. 1, the FFT spectrum of oscillations in a single island, i.e., every fourth turn around the ring for the fourth-order resonance, is shown in Fig. 3. Note that there are two dominant peaks: one located at $\nu_{\rm coupling}=\nu_x-\nu_y+1=0.0524\pm0.0007$ due to linear coupling and another corresponding to the island tune $\nu_{\rm island}=0.0013\pm0.0007$. The accuracy of the island tune measurement is limited by the available memory in the transient recorders. Without linear coupling, the particle would have completed one oscillation around an island's fixed point after $1/\nu_{\rm island}$ orbital revolutions. The small-amplitude oscillation around the island fixed point is also an ellipse.

The one-dimensional resonance island ellipse shown in the right-hand side of Fig. 1 is obscured by the linear coupling. Yet the island structure is retained. The motion is a superposition of the more rapid coupling oscillation and the slower resonance island oscillation. The phase-space trajectory appears as the coupling oscillation winding around a resonance island ellipse (see Fig. 4). For the coupling tune of 0.0524 at the fourth-order betatron resonance condition, it takes five island turns (e.g., the first, fifth, ninth, thirteenth, and seventeenth orbital turns for the first island, etc.) for the particle to complete one loop around a centroid in the coupling ellipse. The size of the coupling loop depends on the betatron coupling strength. A five-island-turn moving average of the phase-space coordinates will effectively eliminate this rapid coupling motion, revealing the slower resonance island oscillation. The moving average will trace out an ellipse around the stable fixed point of an island with a characteristic frequency of the island tune $\nu_{\rm island}=0.013$, which corresponds to a period of over 800 orbital turns or 200 island turns.

Near the single resonance, $m\nu\approx n$, the Hamiltonian can be approximated by [1] $H=H_0(J)+g(J)\cos(m\phi-n\theta-\chi)$. Here (J,ϕ) are the conjugate action-angle variables of the betatron motion, and χ is a phase factor determined by the distribution of nonlinear elements in the accelerator. The betatron tune is given by $\nu(J)=\partial H/\partial J\approx\nu_0+\alpha J$, where we have used a first-order Taylor series expansion in the action variable with ν_0 as the betatron tune at zero betatron amplitude and α the coefficient of the first-order expansion. $g(J)$ is related to the resonance strength and $\theta=s/R$ is the orbital angle around an accelerator. For the present study, $m=4$ and $n=15$.

A canonical transformation with generating function $F_2(\phi,J_1)=[\phi-(n/m)\theta]J_1$ can be performed easily to yield a new Hamiltonian, $\tilde H=H_0(J_1)-(n/m)J_1+g(J_1)\times\cos(m\phi_1-\chi)$, where (J_1,ϕ_1) are the new conjugate action-angle variables with $J_1=J$. Note here that the new Hamiltonian $\tilde H$ is a constant of motion; the particle trajectory follows a constant contour of $\tilde H$. Fixed points of the Hamiltonian are given by $\partial\tilde H/\partial J_1=0$ and $\partial\tilde H/\partial\phi_1$

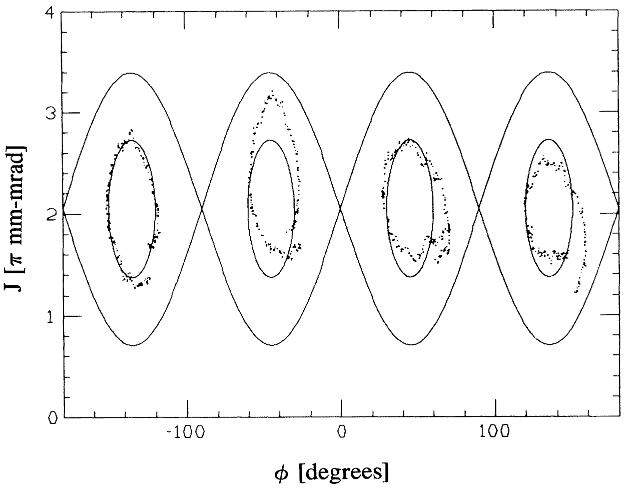

FIG. 5. The stable ellipse around island fixed points in the action-angle variable is fitted by the Hamiltonian of Eq. (2) with $\nu_{\text{island}}=0.0013$, obtained from the FFT analysis. The action and angle variables are obtained from averaging every five island turns in each island in order to eliminate the effect of the coupling resonance. We found $a=0.00048\pm0.0001$ (π mm mrad)$^{-1}$.

$=0$, i.e., $\nu_1(J_1)-n/m+g'(J_1)\cos(m\phi_1-\chi)=0$ and $\sin(m\phi_1-\chi)=0$.

Let J_r be the corresponding action such that the betatron tune satisfies a resonance condition, i.e., $m\nu(J_r)=n$. The Hamiltonian can then be expanded around the resonant action:

$$\tilde{H}=\tfrac{1}{2}a(J_1-J_r)^2+g(J_r)\cos(m\phi_1-\chi)+\cdots . \quad (2)$$

Thus the equation of motion in the resonance region satisfies the pendulumlike equation of motion. The island tune is given by $\nu_{\text{island}}=m|ag|^{1/2}$. Hence the resonance strength is given by $g=\nu_{\text{island}}^2/m^2a$. The island width, or the maximum difference in the action variables between the stable fixed point and the separatrix, is given by $\Delta J=J_1-J_r\approx 2[g(J_r)/a]^{1/2}=2\nu_{\text{island}}/ma$.

The parameter a could be obtained from the slope of the betatron tune as a function of the kicked betatron amplitude J. Alternatively, the ellipses of particle motion around the stable fixed point can be described by the invariant Hamiltonian of Eq. (2). Substituting $g=\nu_{\text{island}}^2/m^2a$ into Eq. (2), the parameter a can be obtained through matching the particle trajectory with the contour of the Hamiltonian. Figure 5 shows a (J,ϕ) plot of the data of the island ellipses of Fig. 1 after taking a five-island-turn moving average in each island in order to remove the coupling motion. Using the Hamiltonian in Eq. (2), we obtain $a=0.00048\pm0.0001$ (π mm mrad)$^{-1}$. The corresponding separatrix is also shown in Fig. 5.

In conclusion, we studied properties of fourth-order nonlinear resonance islands. One interesting feature is that the betatron coupling does not destroy the structure of one-dimensional resonance islands. Experimental data were used to determine resonance island parameters, ν_{island}, J_r, and a. The Hamiltonian for the particle motion was derived near the resonance region for the first time. Using the experimentally derived Hamiltonian, a more reliable prediction of particle motion may be possible. These experimental nonlinear-beam-dynamics studies may prove to be useful in an effort to understand the dynamical aperture and the long-term behavior of particle motion for future colliders, such as the SSC and RHIC.

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