Exact Quantum-Statistical Dynamics of an Oscillator with Time-Dependent Frequency and Generation of Nonclassical States

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Exact results on the time evolution of the density matrix and the Wigner function for an oscillator with time-dependent frequency are given. Explicit time dependence is given for the case of linear sweep of the restoring force. Such a sweep is shown to generate states of the oscillator with remarkable nonclassical properties such as squeezing and sub-Poissonian character.

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The generation of nonclassical states [1] of the radiation field and other systems continues to be of great interest. One now knows a very large class of nonlinear optical processes which can produce squeezed light and some of these schemes have already been demonstrated. We consider here a linear system with externally controllable parameters.

In this Letter we demonstrate how states with nonclassical properties can be generated if the frequency of the harmonic oscillator is swept as a function of time. We present *exact* analytical results for the case of a linear sweep of the restoring force. We present results not only for some of the lower-order moments like $\langle \hat{x}^2(t)\rangle$, $\langle \hat{p}^2(t)\rangle$, and number fluctuations but also give the time evolution of the density matrix of the system. Such linear systems also describe the quantum motion of particles in a Paul trap [2–4]. Thus the general results on the time evolution of the density matrix and the Wigner function would also be applicable to particles in Paul traps.

Consider a harmonic oscillator with a time-dependent frequency [5]. We write the Hamiltonian in the form

$$\hat{H} = \tfrac{1}{2}\hat{p}^2 + \tfrac{1}{2}\omega^2\hat{x}^2[1+\beta(t)]\,, \tag{1}$$

where $\beta(t)$ gives the frequency modulation. In our analytical work we assume that the restoring force on the oscillator is linearly swept, i.e.,

$$\beta(t) = \begin{cases} 0, & \text{for } -\infty < t < 0\,, \\ \beta_0 t/T, & \text{for } 0 \le t \le T\,, \\ \beta_0, & \text{for } T < t < \infty\,. \end{cases} \tag{2}$$

Other forms of $\beta(t)$ are also very important—for example, in a Paul trap $\beta(t)$ is equal to $\beta_0\cos(\Omega t)$, i.e., it is periodic. The two limiting cases corresponding to sudden and adiabatic changes are contained in (2).

We next show how the density matrix at time t can be obtained from the initial density matrix. We introduce the scaled operators $\hat{X}$ and $\hat{P}$ and dimensionless time τ

$$\hat{X} = \sqrt{\omega/2}\,\hat{x}, \quad \hat{P} = \sqrt{1/2\omega}\,\hat{p}\,, \qquad [\hat{X},\hat{P}] = i/2, \quad \tau = \omega t, \quad \hbar = 1\,. \tag{3}$$

The Heisenberg equations for $\hat{X}$ and $\hat{P}$ can be solved and the solution can be written in the form

$$\begin{pmatrix} \hat{X}(\tau) \\ \hat{P}(\tau) \end{pmatrix} = \begin{pmatrix} U & V \\ \dot{U} & \dot{V} \end{pmatrix} \begin{pmatrix} \hat{X}(0) \\ \hat{P}(0) \end{pmatrix}, \tag{4}$$

where the functions U and V satisfy the differential equation

$$\frac{d^2}{d\tau^2}\phi + [1+\beta(\tau)]\phi = 0 \tag{5}$$

with the initial conditions

$$U(0)=1, \quad \dot{U}(0)=0\,, V(0)=0, \quad \dot{V}(0)=1\,. \tag{6}$$

For $\beta(t)$ given by (2) the general solution of (5) in the domain $0 \le \tau \le \omega T$ can be expressed in terms of Bessel functions of order $\frac{1}{3}$

$$\phi = C_1(\tau')^{1/2}J_{1/3}(z) + C_2(\tau')^{1/2}Y_{1/3}(z)\,, \tag{7}$$

where

$$\tau' = \tau + \frac{\omega T}{\beta_0}, \quad z = \frac{2}{3}\left(\frac{\beta_0}{\omega T}\tau'^3\right)^{1/2}. \tag{8}$$

Thus the functions U and V can be fixed by using (6) in (7). Note that for $\tau = 0$, $z \neq 0$ and thus both C_1, C_2 will be nonzero. The solution of the Heisenberg equations in the domain $\tau > \omega T$ will be

$$\begin{pmatrix} \hat{X}(\tau) \\ \hat{P}(\tau) \end{pmatrix} = \begin{pmatrix} \cos[k(\tau-\omega T)] & (1/K)\sin[k(\tau-\omega T)] \\ -k\sin[k(\tau-\omega T)] & \cos[k(\tau-\omega T)] \end{pmatrix} \begin{pmatrix} \hat{X}(\omega T) \\ \hat{P}(\omega T) \end{pmatrix}, \tag{9}$$

where $k^2 = 1+\beta_0$.

Equations (4)–(9) give the exact solution of the Heisenberg equations of motion. We next use these to study the quantum dynamics of the oscillator. The functions U and V depend on how the frequency is swept. We can also define the creation and annihilation operators in the usual way,

$$\hat{a} = \hat{X} + i\hat{P}, \quad \hat{a}^\dagger = \hat{X} - i\hat{P}, \quad [\hat{a},\hat{a}^\dagger] = 1\,. \tag{10}$$

It is clear from the foregoing that the time dependence of $\hat{a}$ is of the form

$$\hat{a}(\tau) = u(\tau)\hat{a} + v(\tau)\hat{a}^\dagger, \quad |u|^2 - |v|^2 = 1\,. \tag{11}$$

The parameters $u(\tau)$ and $v(\tau)$ can be obtained from (4) to (9). The Bogoliubov transformation (11) with time-dependent parameters enables us to obtain the density matrix at time τ in terms of the density matrix at time $\tau=0$. Using disentangling theorems for the SU(1,1) group we have proved that

$$\hat{a}(\tau)=\hat{S}^{-1}\hat{a}\hat{S},\quad \hat{\rho}(\tau)=\hat{S}\hat{\rho}(0)\hat{S}^{-1},$$
$$\hat{S}=\exp(i\theta_u\hat{a}^\dagger\hat{a})\exp\left(\tfrac{1}{2}e^{+i(\theta_v-\theta_u)}\cosh^{-1}|u|\hat{a}^{\dagger 2}-\text{H.c.}\right), \tag{12}$$

where $u=|u|e^{i\theta_u}$, $v=|v|e^{i\theta_v}$. It should be borne in mind that θ_u, θ_v, and $|u|$ are functions of time τ and that in general $\theta_u\neq 0$. Note that the standard squeezing operator [6] corresponds to choosing $\theta_u=0$.

Instead of working with the density matrix, we could also work directly with the Wigner function [7] which is very useful in the study of quantum-statistical properties. It is in fact much simpler to write the Wigner function $\Phi(\alpha,\alpha^*)$ at time τ in terms of the Wigner function at $\tau=0$. Using the transformation (11) one can prove that [cf. Ref. [7(b)], Eq. (2.22)]

$$\Phi(\alpha,\alpha^*,\tau)=\Phi(u^*\alpha-v\alpha^*;u\alpha^*-v^*\alpha;0). \tag{13}$$

A very large class of states [7] of the harmonic oscillator corresponds to the Gaussian Wigner function of the form

$$\Phi(\alpha,\alpha^*)=\frac{1}{\pi(\gamma^2-4|\mu|^2)^{1/2}}\exp\left(-\frac{\mu(\alpha-\alpha_0)^2+\mu^*(\alpha^*-\alpha_0^*)^2+\gamma|\alpha-\alpha_0|^2}{\gamma^2-4|\mu|^2}\right), \tag{14}$$

where

$$\langle\hat{a}\rangle=\alpha_0,\quad \langle\hat{a}^2\rangle=-2\mu^*+\alpha_0^2,\quad \langle\hat{a}^\dagger\hat{a}\rangle=\gamma-\tfrac{1}{2}+|\alpha_0|^2. \tag{15}$$

The form (14) includes (a) coherent states $\mu=0$, $\gamma=\frac{1}{2}$, (b) thermal states $\mu=0$, $\alpha_0=0$, $\gamma>\frac{1}{2}$, (c) mixtures of thermal and coherent states, and (d) squeezed states $\mu\neq 0$, $\alpha_0\neq 0$, $\gamma^2-4|\mu|^2=\frac{1}{4}$. From the transformation (13) it is clear that the Wigner function at time τ will also be Gaussian with the parameters μ, γ, and α_0 replaced by

$$\alpha_0\to u(\tau)\alpha_0+v(\tau)\alpha_0^*,$$
$$-2\mu^*\to u^2(-2\mu^*)+v^2(-2\mu)+2uv\gamma\equiv-\tfrac{1}{2}\eta\sinh\Lambda e^{i\theta_\mu}, \tag{16}$$
$$\gamma\to(u^*u+v^*v)\gamma-2\mu u^*v-2\mu^*uv^*\equiv\tfrac{1}{2}\eta\cosh\Lambda.$$

The quadrature $e^{-i\chi}\hat{a}+e^{i\chi}\hat{a}^\dagger$ will exhibit squeezing if

$$\eta[\cosh\Lambda-\sinh\Lambda\cos(\theta_\mu-2\chi)]<1. \tag{17}$$

If the phase of μ is nonzero, then the quadratures $\hat{X}$ and $\hat{P}$ are correlated. The Gaussian character of the Wigner function can also be used to calculate the width of the photon number distribution. The variance of the number distribution or Mandel's Q parameter can be written in terms of μ and γ as follows:

$$Q\equiv\frac{\langle\hat{n}^2\rangle-\langle\hat{n}\rangle^2-\langle\hat{n}\rangle}{\langle\hat{n}\rangle}$$
$$\equiv\left(\frac{\gamma^2+2|\alpha_0|^2\gamma-\frac{1}{4}-2(\alpha_0^*)^2\mu^*-2\alpha_0^2\mu+4|\mu|^2}{\gamma+|\alpha_0|^2-\frac{1}{2}}\right)-1. \tag{18}$$

The *transition probabilities* can be directly obtained from (12). For example, the probability p_{nm} of finding the oscillator in the state $|n\rangle$ at time τ given that at $\tau=0$ it was in the state $|m\rangle$ is

$$p_{nm}(\tau)=|\langle n|\hat{S}|m\rangle|^2. \tag{19}$$

The matrix element of the operator $\hat{S}$ can be obtained by expressing $\hat{S}$ in normally ordered form.

It should be borne in mind that all the above results (10)-(19) hold *irrespective* of the form of time-dependent function $\beta(t)$. *Explicit* results can be obtained for the case of *linear sweep* of the restoring force by using (4) to (9).

We next present the numerical results for the squeez-

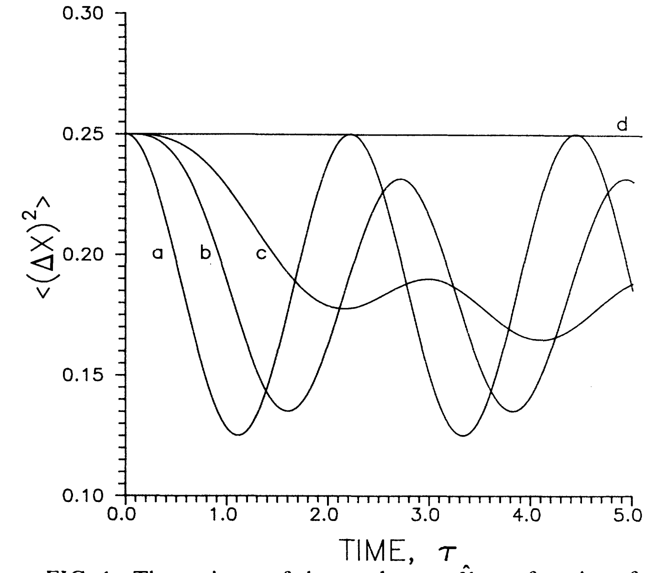

FIG. 1. The variance of the quadrature $\hat{X}$ as a function of τ for an oscillator initially in the ground state. The parameters are chosen as $\beta_0=1$ and (curve *a*) $\omega T=10^{-3}$, (*b*) 1, (*c*) 3, and (*d*) 10^3. Thus the cases *a* and *d* correspond, respectively, to sudden and adiabatic limits.

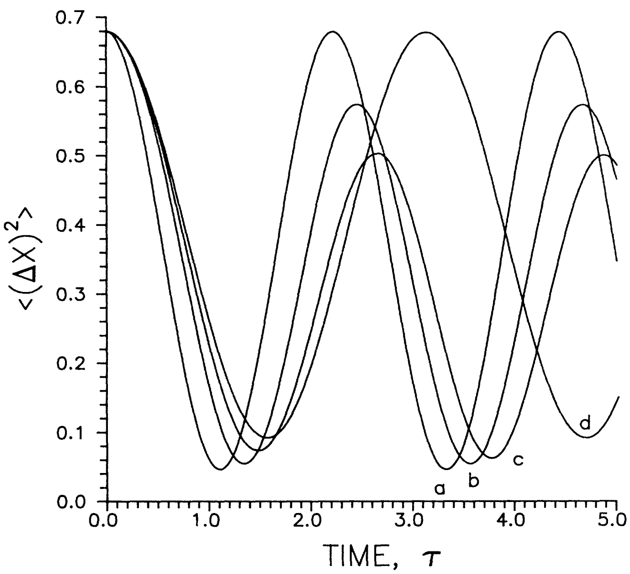

FIG. 2. Same as in Fig. 1 but the oscillator is initially in a squeezed coherent state $|\alpha,\zeta\rangle$ with $\alpha=1$, $\zeta=0.5e^{-i\pi}$.

ing and sub-Poissonian properties of the oscillator system. In Fig. 1 we show the squeezing in the component $\hat{X}=(\hat{a}+\hat{a}^\dagger)/2$ when initially the oscillator is in the ground state. We observe that the linear sweep produces a *significant amount of squeezing*. The squeezing properties are much more *pronounced* for the case of a *sudden jump* [8]. As expected the adiabatic changes [9] do not produce any noticeable squeezing. From the calculation of the phases θ_u and θ_v we also find that the two quadratures $\hat{X}$ and $\hat{P}$ are in general correlated for most of the time. Note that for fast sweeping, the variance exhibits periodic behavior. For the parameters of Fig. 1 this period is found to be $\pi/\sqrt{2}$ which follows from Eq. (5) as $1+\beta \rightarrow 2$. In Fig. 2 we show the squeezing characteristics if initially the oscillator state is squeezed in the quadrature $\hat{P}=(\hat{a}-\hat{a}^\dagger)/2i$. The quadrature $\hat{X}$ exhibits quite a significant amount of squeezing which in turn depends on the rate of the frequency sweep. For the initial vacuum state the Wigner function (14) is Gaussian with equal noise in the two quadratures ($\mu=0$, $\gamma=\frac{1}{2}$). In Fig. 3 we show the time evolution of the Wigner function (14). We show the behavior at a time when the system shows the

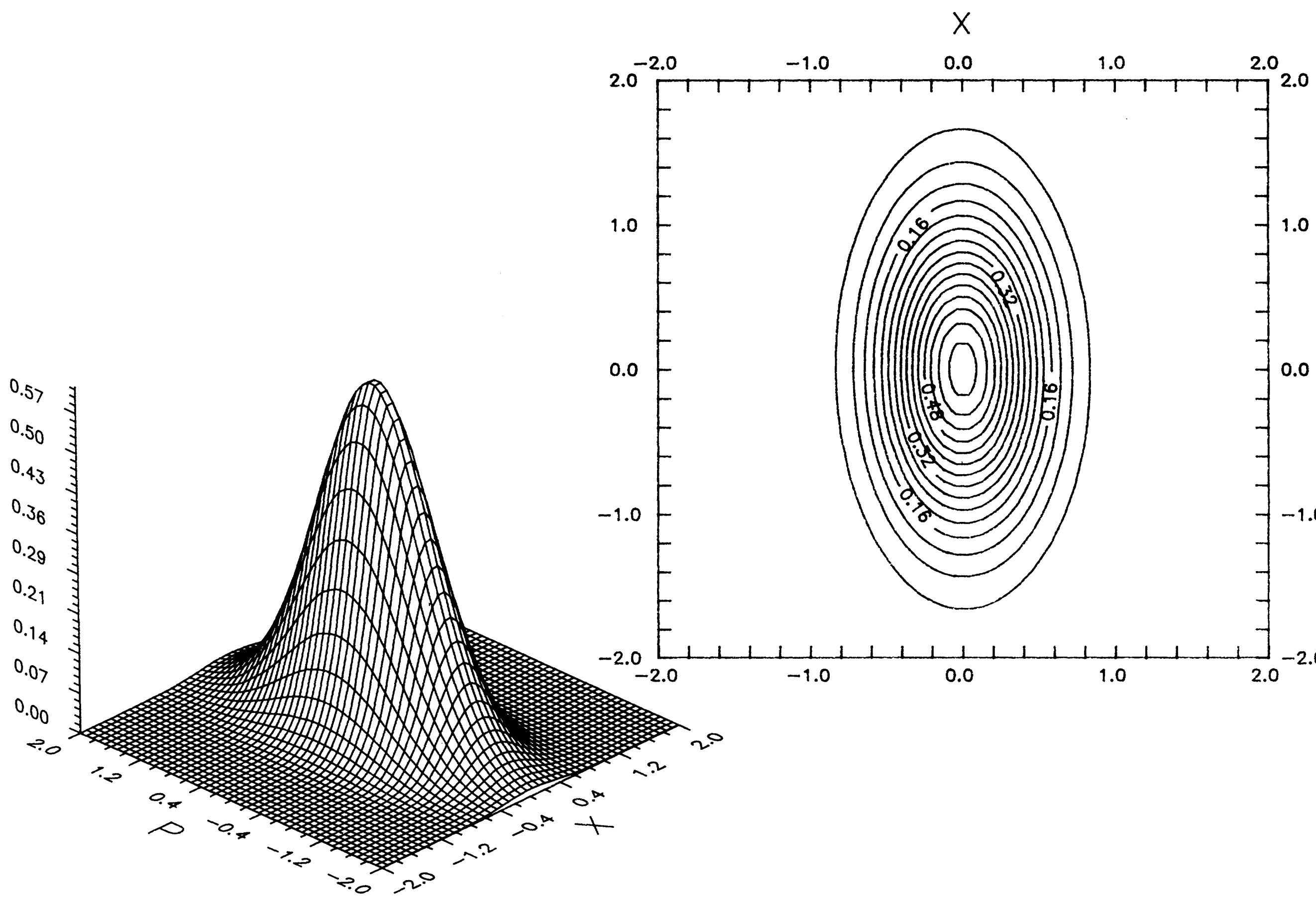

FIG. 3. The Wigner function (14) with $\alpha=X+iP$ [cf. Eq. (10)] for the system initially in vacuum state for $\omega T=10^{-3}$, $\beta_0=1$, and $\tau=1.1$ which corresponds to the minimum in Fig. 1, curve *a*. Inset: The contours of constant Φ.

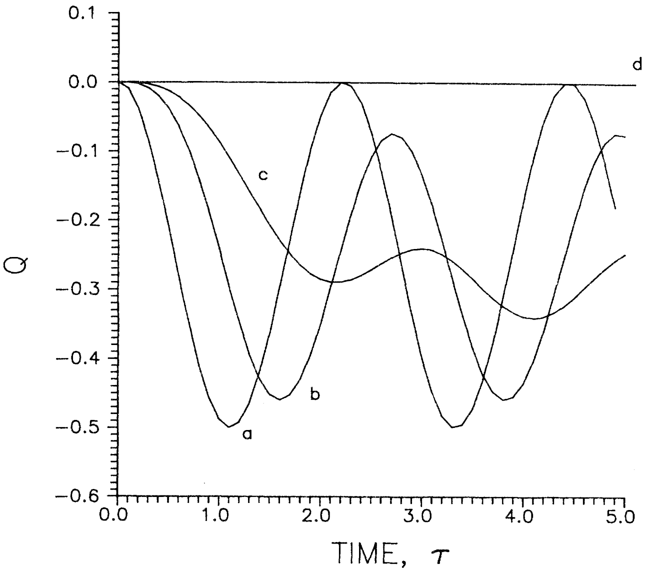

FIG. 4. The Q parameter as a function of τ for the case of an oscillator initially in a coherent state $|\alpha\rangle$, $\alpha=1$. The parameter ωT has been chosen as (curve a) 10^{-3}, (b) 1, (c) 3, and (d) 10^3.

maximum amount of squeezing in the $\hat{X}$ quadrature. Finally in Fig. 4 we show the generation of sub-Poissonian statistics when initially the state is a coherent state [10]. The time-dependent behavior of the Q parameter is similar to that shown in Fig. 1. In general this is not expected except when the mean value of the field is so large [11] that a linearization around the mean value can be done. For Fig. 1, the mean value is zero but this is not so for Fig. 4. The linear sweep of the restoring force can produce *large amounts of sub-Poissonian statistics* [12].

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