PHYSICAL REVIEW LETTERS

VOLUME 67 | 23 DECEMBER 1991 | NUMBER 26

Metamorphosis of a Cantor Spectrum Due to Classical Chaos

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(Received 25 June 1991)

We study how a Cantor spectrum, its level statistics, and corresponding dynamics are affected by the onset of classical chaos. While the spectrum undergoes visible changes, its level spacing distribution follows an inverse power law $p(s) \sim s^{-3/2}$ on small scales. We find a crossover which is manifested in the time domain by two diffusive regimes characterized by a classical and a quantum-mechanical diffusion coefficient. In the strong quantum limit we show by means of a transformation that the spectrum is governed by the *integrable* Harper equation, even if the classical phase space is strongly chaotic.

PACS numbers: 05.45.+b, 03.65.−w, 73.20.Dx

The impact of classical chaos on quantum mechanics is of interest both from an experimental and a theoretical point of view [1]. In a search of signatures for classical chaos the statistics of levels is frequently investigated [2]. The distribution $p(s)$ of nearest-neighbor level spacings exhibits power laws $p(s) \sim s^{\beta}$ ($\beta = 1,2,4$) for $s \to 0$ as a consequence of level repulsion. We recently pointed out another class of level statistics characterized by an *inverse* power law $p(s) \sim s^{-3/2}$, which appears to be universal for systems with unbounded quantum diffusion in one dimension [3]. We found a realization of this unbounded diffusion in a model of Bloch electrons in a magnetic field as described by Harper's equation [4–7]. The classical limit of this model, however, is an integrable system and shows no chaotic behavior, whereas the real classical analog of crystal electrons in a magnetic field (e.g., in lateral surface superlattices) should have diffusive chaotic dynamics [8]. As Hofstadter's butterfly spectrum [6] stems from a classically integrable model (Harper's equation) and as the long-standing goal of its experimental observation is now approached with lateral surface superlattices [9], it is of considerable experimental interest to know if such spectra are changed significantly by the existence of classical chaos.

While the influence of classical chaos on discrete spectra and their level statistics was studied intensely in the past, we now investigate how a Cantor spectrum, i.e., an uncountable spectrum, is affected. To this end we assume the kicked Harper model as a simple classically chaotic modification of Harper's equation [10]. We find that the asymptotic power law $p(s) \sim s^{-3/2}$ still accounts for the classically chaotic case. A mere inspection of the spectrum, on the other hand, reveals considerable changes. The discrepancy is explained by a crossover on a scale s^* and on a corresponding time scale t^* depending on $\hbar$. On small energy scales $s < s^*$ the spectral statistics shows hierarchical level clustering $p(s) \sim s^{-3/2}$ unaffected by classical chaos. The level clustering is destroyed only on scales $s > s^*$. In the time domain there is a corresponding crossover from a mimicking of the chaotic classical diffusion below t^* to a purely quantum mechanical diffusion above t^*. In the strong quantum limit ($\hbar \to \infty$) we show that the system can be transformed to the integrable Harper system. Thereby we explain why the spectrum has the statistics of the integrable Harper system, even if the classical phase space is strongly chaotic.

The unperturbed Harper's equation [4–7] is a stationary Schrödinger equation $\hat{H}\psi = E\psi$, which can formally be expressed in terms of a Hamiltonian

$$\hat{H} = 2\cos(\hat{p}) + \lambda\cos(\hat{x}) . \tag{1}$$

Here $\lambda = 2$ and $\hat{p} = -i\hbar\, \partial/\partial x$ plays the role of a momentum operator with an effective $\hbar = 2\pi\sigma$. The parameter σ is proportional to the magnetic field and gives the number of flux quanta per unit-cell area. Note that the classical limit of the Hamiltonian Eq. (1) is integrable. We want to study the above questions in the kicked Harper model

[11,12]

$$\hat{H} = L\cos(\hat{p}) + K\cos(\hat{x})\delta_1(t)\,, \tag{2}$$

where $\delta_1(t)$ is a periodic delta function of period one. The corresponding classical equation of motion is an iterated map from one kick to the next

$$p' = p + K\sin(x), \quad x' = x - L\sin(p')\,. \tag{3}$$

In Fig. 1 we show typical Poincaré surfaces of section for four different values of $K = L \equiv \kappa$. For $\kappa \rightarrow 0$ the unkicked Harper system is recovered. As soon as $\kappa > 0$ a small stochastic layer exists, which widens up with increasing kicking strength and allows for diffusive motion in p and x. In an intermediate regime there is a mixed phase space with regions of regular and stochastic motion. For $\kappa = 5$ no islands are visible and the dynamics is strongly chaotic.

The quantum dynamics of Eq. (2) is described by the time-dependent Schrödinger equation. As the kicks are periodic in time, the Floquet theorem applies and one can use quasienergy eigenstates $\Psi_\omega(x,t)$ [13]. The system is also periodic in x, which requires the Bloch theorem to hold, i.e., $\Psi_\omega(x,t) = e^{i\theta_x x}\psi_\omega(x,t)$ where θ_x is the quasimomentum in the x direction and $\psi_\omega(x+2\pi,t) = \psi_\omega(x,t)$. The periodicity of $\psi_\omega(x,t)$ now implies that the operator $\hat{p}$ has eigenvalues of the form $\hbar(n+\theta_x)$ with integer n. The time evolution operator acting on periodic functions in x for one period of time is then given by

$$\hat{U} = \exp\{-i(L/\hbar)\cos[\hbar(\hat{n}+\theta_x)]\}\exp[-i(K/\hbar)\cos(x)] \tag{4}$$

and determines the quasienergies ω by $\hat{U}\psi_\omega(x,t) = e^{i\omega}\psi_\omega(x,t)$. The degree of classical stochasticity and quantum effects can be varied independently by varying K, L, and $\hbar$, respectively.

We have investigated the integrated level spacing distribution and the time evolution of the variance

$$\mathrm{var}(t) = \hbar^2\langle n^2(t)\rangle = \hbar^2\sum_n n^2|\phi_n(t)|^2 \tag{5}$$

of a wave packet $\phi(t)$ for different values of κ and $\hbar = 2\pi\sigma$ in the kicked Harper model. For the time evolution of a wave packet the operator $\hat{U}$ was iterated using the fast-Fourier-transform method [14] with up to 10^5 momentum eigenstates. We have analyzed the variance for initial Gaussian wave packets with equal width in position and momentum. The spectrum was studied for rational values [15] of $\sigma = r/q$. Here the system is periodic in x and p and another Bloch phase θ_p is introduced. This reduces the time-evolution operator in Eq. (4) to a $q \times q$ matrix [16]. A Fourier transform of the time evolution of a wave packet yields those quasienergies $\omega \in [-\pi,\pi]$ which belong to eigenstates excited by the wave packet. To obtain all q quasienergies it turns out to be most efficient to start with a random initial wave packet. The quasienergies ω depend on the two phases θ_p and θ_x and form energy bands, as long as σ is rational.

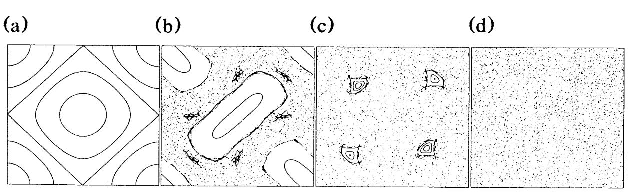

FIG. 1. Classical phase space of one unit cell for $\kappa = 0.01$, 2.0, 3.5, and 5.0 (left to right).

We first study how Hofstadter's butterfly is affected by classical chaos, i.e., the spectrum (of quasienergies in our case) as a function of σ. In Fig. 2 these bands are plotted as functions of σ for four different values of $\kappa/\hbar$. As can be seen from Eq. (4), for $\kappa/\hbar =$const, the spectrum is symmetric with respect to $\sigma = \frac{1}{2}$. For $\kappa/\hbar = 1$ (top) the spectrum looks like the original Hofstadter butterfly. With increasing values of $\kappa/\hbar$ we observe that the large gaps become smaller and eventually disappear. For small denominators q of σ the q single bands broaden until they touch each other and form one single band from $-\pi$ to π. For σ's close to irrational numbers there is a transition from a hierarchical band clustering to a more uniform distribution of bands all over the interval. Although Fig. 2 exhibits visible changes in the butterfly, the asymptotic form of the level spacing distribution $p(s) \sim s^{-3/2}$ remains unchanged (see below). We get more insight into this behavior by studying time-dependent phenomena. In the integrable Harper model we had found an unbounded linear (i.e., diffusive) spread of wave packets [3]. Subsequently unbounded diffusion was also observed in the classically chaotic kicked Harper model by Lima and Shepelyansky [12]. We now investigate how these observations are related. Figures 3(a) and 3(b) compare the growth of the variance of a quantum-mechanical wave packet and a corresponding classical distribution in phase space for kicking strengths κ corresponding to Figs. 1(c) and 1(d), respectively. The quantum behavior mimics the classical diffusion for a finite time as is seen in the insets. There is a crossover time t^*, after which unbounded quantum diffusion due to hierarchical level clustering like in the integrable case dominates. Thus there are a classical and a quantum-mechanical diffusive regime with different diffusion coefficients and the transition can be detected. This crossover is to be contrasted to the diffusion-localization crossover known from the kicked rotator [17].

For the wave packet of Fig. 3(a) [$\kappa = 3.5$, $\hbar = 2\pi/(400+\sigma_g)$], which was centered initially on the outermost torus of the upper right island, the transition occurs after about 20 time steps. Numerically the crossover time t^* scales roughly as $t^* \sim 1/\hbar$ in this situation. Here the classical diffusion coefficient is reduced due to the fact that many classical trajectories are trapped within the island and do not contribute to diffusion. More typically, however, we find the classical diffusion coefficient

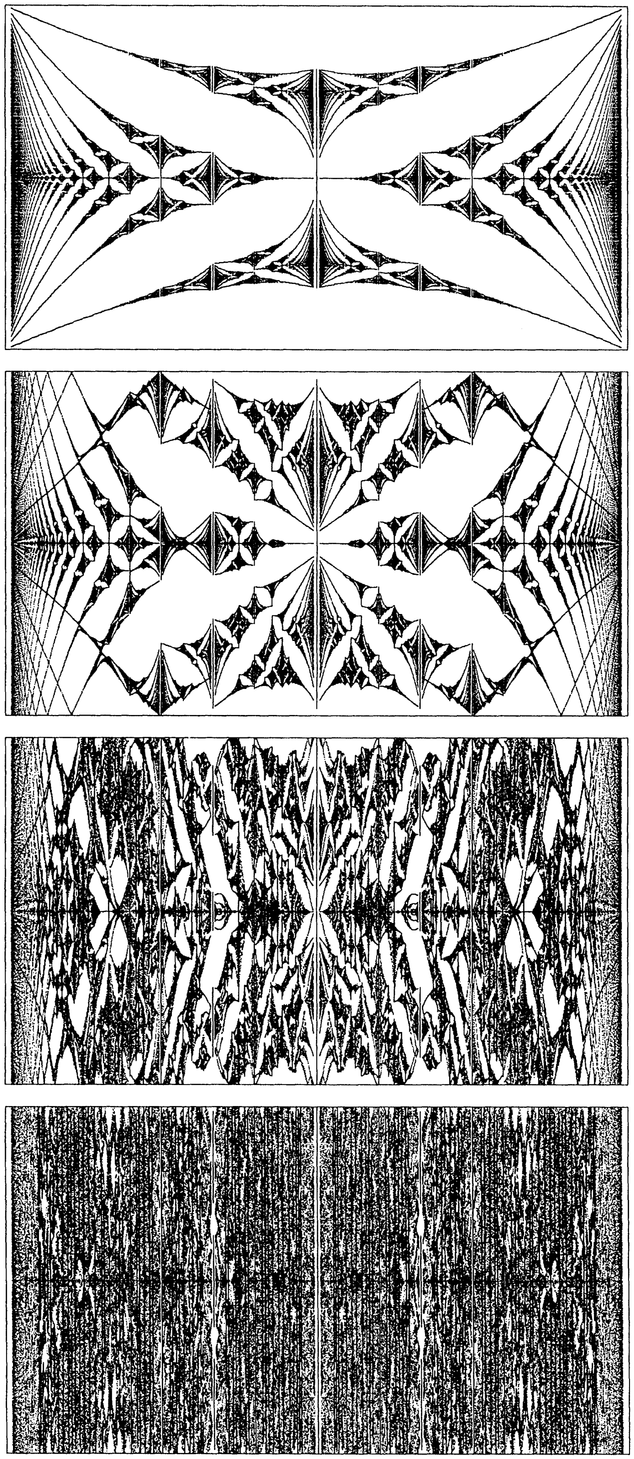

FIG. 2. Quasienergy spectrum vs $\sigma \in [0,1]$ for $\kappa/\hbar = 1$, 3, 6, and 10 (top to bottom). The quasienergies ω range from $-\pi$ to π (-2 to 2 in the top figure). In the Hofstadter butterfly increasing $\kappa/\hbar$ removes the large energy gaps and broadens the bands. The hierarchical level clustering is removed on large scales. Note that we keep $\kappa/\hbar$ constant as $\sigma = 2\pi\hbar$ is varied.

to exceed the quantum-mechanical one. An example is shown in Fig. 3(b) [$\kappa = 5.0$, $\hbar = 2\pi/(10+\sigma_g)$] where the wave packet moves in a classically fully chaotic phase space. In this case the crossover time numerically appears to scale roughly as $\tau \sim 1/\hbar^2$. Thus the fingerprints of classical chaos in the diffusion behavior of the kicked Harper model are the short time diffusion and its scaling. One expects the crossover in the time domain to correspond to a crossover at some scale s^* in the spectrum. For the integrated level spacing distribution (Fig. 4) we find an inverse power law $p_{\rm int}(s) \sim s^{-1/2}$ for small spacings $s < s^*$ [18]. On small scales the spectrum thus is unaffected by classical chaos. The figure clearly shows a crossover on larger scales where the hierarchical level clustering is destroyed due to classical chaos. It was not possible to detect a transition to a Wigner distribution within a single spectrum. For large κ, $p(s)$ becomes a Wigner distribution.

One concludes from the above results that the asymptotic behavior ($t \to \infty$, $s \to 0$) of the kicked system is determined by the *integrable* Harper's equation already. In the strong quantum limit ($K/\hbar, L/\hbar \to 0$) we can explain this by mapping the eigenvalue equation of $\hat{U}$ to the integrable Harper's equation. Denoting by ψ^- and ψ^+ the eigenfunctions before and after the kick, in analogy to

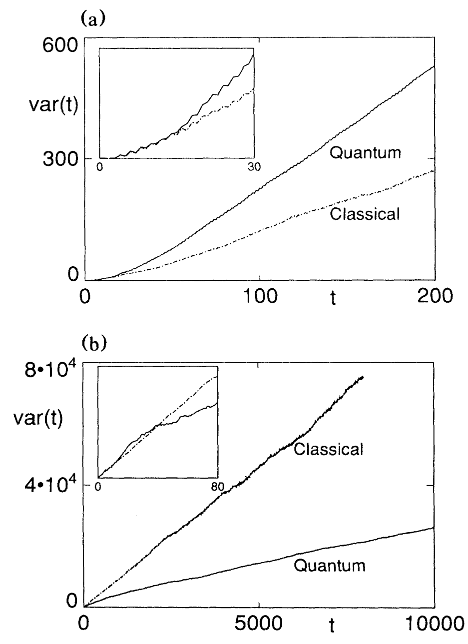

FIG. 3. (a) Quantum vs classical time evolution (solid and dash-dotted lines, respectively) of the variance of a wave packet initially centered on the outermost torus of the upper right island in Fig. 1(c) with $\kappa = 3.5$ and $\hbar = 2\pi/(400+\sigma_g)$. The inset shows that after a crossover time a second diffusion process sets in, i.e., quantum diffusion due to hierarchical level clustering. (b) Same as (a) for the fully chaotic phase space of Fig. 1(d) with $\kappa = 5$ and $\hbar = 2\pi/(10+\sigma_g)$. Here the quantum diffusion coefficient is smaller than the classical.

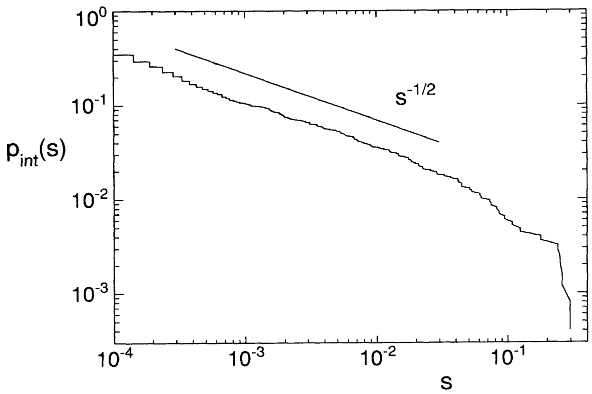

FIG. 4. Integrated level-spacing distribution for $\kappa=5.0$ and $\sigma=233/2474$, an approximant of $1/(10+\sigma_g)$ [corresponding to Fig. 3(b)]. There is an *inverse* power law $p_{\rm int}(s) \sim s^{-1/2}$ on small scales and a crossover to a regime influenced by classical chaos for larger s. The spectrum was determined by a Fourier transform of the time evolution of a wave packet over 2^{17} time steps yielding a quasienergy resolution of about 10^{-4}.

Grempel, Prange, and Fishman [14], we define

$$\bar{\psi}_\omega(x,t) = \psi_\omega^+(x,t)/\{1 - i\tan[(K/2\hbar)\cos x]\} . \qquad (6)$$

We use the Floquet theorem and write $\bar{\psi}_\omega(x,t) = e^{i\omega t}\bar{u}_\omega(x,t)$ where $\bar{u}(x,t+1) = \bar{u}(x,t)$. For the Fourier components $\varphi_n = (2\pi)^{-1}\int_0^{2\pi} e^{inx}\bar{u}_\omega(x,t)dx$ we formally obtain a tight-binding equation

$$\sum_{r(\neq 0)} W_r \varphi_{n+r} + T_n \varphi_n = -W_0 \varphi_n , \qquad (7)$$

where the hopping terms W_r are the Fourier coefficients of $W(x) = -\tan[(K/2\hbar)\cos x]$ and $T_n = \tan\{\omega/2 - (L/2\hbar)\cos[\hbar(n+\theta_x)]\}$. A first-order expansion in $L/\hbar$ and $K/\hbar$ yields

$$\varphi_{n+1} + \varphi_{n-1} + \frac{2L}{K\cos^2\omega/2}\cos[\hbar(n+\theta_x)]\varphi_n = \frac{4\hbar}{K}\tan(\omega/2)\varphi_n . \qquad (8)$$

For $L/\hbar, K/\hbar \to 0$ the quasienergies ω tend to zero and we identify Eq. (8) with Harper's equation for $\lambda = 2L/K$ and $E = 2\hbar\omega/K$. The integrable Harper's equation thus determines the spectrum of the kicked system even in the strongly chaotic regime, provided $\hbar$ is large enough. We believe that for any value of K, L, and $\hbar$ the asymptotic long-time behavior still resembles the integrable case. Reference [3] then would imply a quadratic growth of the variance for $K > L$, unbounded diffusion for $K = L$, and localization for $K < L$. This would be an alternative explanation for corresponding recent observations by Lima and Shepelyansky [12,19]. We conclude by mentioning that in the experimental search for Hofstadter's butterfly in lateral surface superlattices spectral modifications due to classical chaos similar to Fig. 2 must be expected. Dynamical properties like diffusion, however, can indirectly reveal the hierarchical structure of the energy spectrum.

This work was supported by the Deutsche Forschungsgemeinschaft.

[1] For example, *Chaotic Behavior in Quantum Systems. Theory and Application*, edited by G. Casati (Plenum, New York, 1985).

[2] See, e.g., O Bohigas and M. J. Giannoni, in *Mathematical and Computational Methods in Nuclear Physics*, edited by J. S. Dehesa, J. M. G. Gomez, and A. Polls, Lecture Notes in Physics Vol. 209 (Springer, Berlin, 1984), p. 1; F. Haake, *Quantum Signatures of Chaos*, Springer Series in Synergetics Vol. 54 (Springer, Berlin, 1991).

[3] T. Geisel, R. Ketzmerick, and G. Petschel, Phys. Rev. Lett. **66**, 1651 (1991).

[4] P. G. Harper, Proc. R. Soc. London A **68**, 874 (1955).

[5] M. Ya. Azbel', Zh. Eksp. Teor. Fiz. **46**, 429 (1964) [Sov. Phys. JETP **19**, 634 (1964)].

[6] D. R. Hofstadter, Phys. Rev. B **14**, 2239 (1976).

[7] J. Bellissard, in *Operators Algebras and Applications*, edited by D. E. Evans and M. Takesaki (Cambridge Univ. Press, New York, 1988), Vol. 2, p. 49.

[8] T. Geisel, J. Wagenhuber, P. Niebauer, and G. Obermair, Phys. Rev. Lett. **64**, 1581 (1990).

[9] For example, R. R. Gerhardts, D. Weiss, and U. Wulf, Phys. Rev. B **43**, 5192 (1991).

[10] Thereby the direct relation to the physical system of Bloch electrons is lost.

[11] P. Leboeuf, J. Kurchan, M. Feingold, and D. P. Arovas, Phys. Rev. Lett. **65**, 3076 (1990).

[12] R. Lima and D. L. Shepelyansky, Phys. Rev. Lett. **67**, 1377 (1991).

[13] Ya. B. Zeldovich, Zh. Eksp. Teor. Fiz. **51**, 1492 (1966) [Sov. Phys. JETP **24**, 1006 (1967)].

[14] D. R. Grempel, R. E. Prange, and S. Fishman, Phys. Rev. A **29**, 1639 (1984); D. L. Shepelyansky, Phys. Rev. Lett. **56**, 677 (1986).

[15] As usual in the unperturbed case, irrational values of σ can only be treated via rational approximants. Convergence of physical properties is required and is found numerically.

[16] S.-J. Chang and K.-J. Shi, Phys. Rev. A **34**, 7 (1986).

[17] G. Casati, B. V. Chirikov, F. M. Izrailev, and J. Ford, in *Stochastic Behavior in Classical and Quantum Hamiltonian Systems*, edited by G. Casati and J. Ford, Lecture Notes in Physics Vol. 93 (Springer, Berlin, 1979), p. 344; B. V. Chirikov, F. M. Izrailev, and D. L. Shepelyansky, Sov. Sci. Rev. Sect. C **2**, 209 (1981); S. Fishman, D. R. Grempel, and R. E. Prange, Phys. Rev. Lett. **49**, 509 (1982).

[18] Here $p_{\rm int}(s)$ is the integral of $p(s')$ from s to infinity. As we analyze the spacing between bands, $p(s)$ actually describes gap statistics. In the incommensurate limit, however, the ratio of the bandwidth to gap width shrinks to zero.

[19] Even some exceptions reported in Ref. [12] for $K < L$ turned out to conform with our conjecture after sufficiently long numerical iteration. Besides, for $\hbar > 2\pi$ the system is identical to one with $\hbar' = \hbar(\text{mod}2\pi)$, $K' = K\hbar'/\hbar$, and $L' = L\hbar'/\hbar$.