## Fractal Geometry of Isoscalar Surfaces in Turbulence

In a recent Letter [1], Constantin, Procaccia, and Sreenivasan predict that isothermal surfaces of a passive temperature field $T(\mathbf{x})$ advected by high-Reynolds-number turbulence have a fractal dimension $2<D<3$ within a sphere whose radius is an inertial-range length, but dimension $D=2$ within a sphere of radius small compared to a dissipation length scale. The prediction is nicely supported by an elegant laboratory experiment. I wish to emphasize that this interesting result is independent of what is usually termed intermittency of the turbulence or the temperature field. Indeed it is true already for an isotropic, homogeneous, multivariate Gaussian temperature field whose wave-number spectrum has a power-law form $\propto k^{-n}$ $(1<n<3,\ k_0 \ll k \ll k_d)$, where $k_0$ is a macroscale wave number and $k_d$ is a dissipation cutoff wave number.

The central tool used in [1] is the co-area formula, which yields

$$\int_V |\nabla T| d^3x = \int_{T_{\min}}^{T_{\max}} A_V(T) dT . \tag{1}$$

Here $T_{\min}$ and $T_{\max}$ are the least and greatest temperatures within the volume $V$ and $A_V(T)$ is the total area, within $V$, of the isothermal surface at level $T$. Let $V$ be a sphere of radius $r$. Whatever the value of $r$, the typical value of $|\nabla T|$ within $V$ is

$$|\nabla T|_{\text{typ}} = O(T_0 k_0^{(n-1)/2} k_d^{(3-n)/2}) ,$$

where $T_0$ is the root-mean-square (rms) temperature over the Gaussian field. Therefore the left-hand side of (1) is $\propto r^3$. If $r \ll 1/k_d$, the typical value of $\delta T \equiv T_{\max} - T_{\min}$ is $O(|\nabla T|_{\text{typ}} r)$. Then, $A_V(T) \propto r^3/r = r^2$. But if $1/k_d \ll r \ll 1/k_0$, the typical value of $\delta T$ is $O(|\nabla T|_{\text{typ}} k_d^{(n-3)/2} \times r^{(n-1)/2})$. Thus $A_V(T) \propto r^3/r^{(n-1)/2} = r^{(7-n)/2}$ and

$$D = (7-n)/2 . \tag{2}$$

If $n=\frac{5}{3}$, the classical Kolmogorov value, Eq. (2) gives $D=\frac{8}{3}$.

Dynamics have not been used at all to obtain (2). Suppose now that the rms value of the advecting velocity field within an octave wave-number band centered on $k$ is $v_k \propto k^{-\zeta}$ $(k_0 \ll k \ll k_d,\ 0<\zeta<1)$. The temperature variance contained in this band is $\Psi_k \propto k^{1-n}$. According to Kolmogorov-like phenomenology, the eddy-damping factor acting on temperature fluctuations at $k$ is $\gamma_k \propto v_k k$. Conservative cascade of temperature variance to higher $k$ then requires $\gamma_k \Psi_k = \text{const}$, so that $n=2-\zeta$. Thus (2) implies $D=2.5+\zeta/2$, in agreement with Ref. [1]. Now $n$ is further restricted to $1<n<2$.

It is of interest that the values of $D$ estimated from the experimental data agree with a Gaussian model. If one makes the approximation that the temperature field is Gaussian, a much more detailed experimental test than comparison of exponents is possible. Absolute measurements of the wave-number spectrum of $T$ can be used to calculate statistics of the left-hand side of (1) and of $\delta T$. The results provide statistics of $A_V(T)$ that could be compared directly with absolute measurements of areas at each value of $r$. The Reynolds number need not be large if actual values are the objective, rather than asymptotic exponents.

I have enjoyed discussions with P. Constantin and K. R. Sreenivasan. This work was supported by the National Science Foundation under Grant No. ATM-9021627 and by the Department of Energy under Grant No. DE-FG03-90ER14118.

Robert H. Kraichnan
303 Potrillo Drive
Los Alamos, New Mexico 87544

Received 26 September 1991
PACS numbers: 47.25.Cg, 05.40.+j

[1] P. Constantin, I. Procaccia, and K. R. Sreenivasan, Phys. Rev. Lett. **67**, 1739 (1991).

 