Quantum Vortex Configurations in Three Dimensions

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We examine the kinematics of quantum vortex configurations in three-dimensional space for an ideal incompressible fluid. The permitted quantum configurations are shown to be *surfaces* of vorticity, characterized topologically not only by their knottedness, but also by additional global properties such as twists and holes. Our results, obtained by representing an infinite-dimensional group of diffeomorphisms, contradict conclusions reached by approximating the infinite-dimensional system with finitely many degrees of freedom.

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It is of interest to understand what kinds of extended classical objects can be quantized. Here we investigate this question in the framework of an ideal incompressible fluid [1]. A classical phase-space description of such a fluid [2] can be given in terms of its momentum density $\mathbf{A}(\mathbf{x})$, or equivalently its vorticity density $\mathbf{\Omega}(\mathbf{x})=(1/m)(\nabla\times\mathbf{A})(\mathbf{x})$, where m is a unit mass. However, the uncertainty principle does not permit quantization of every classically allowed vorticity distribution. A "quantum vortex configuration" is a distribution of classical vorticity for which a corresponding quantum version exists, i.e., for which the appropriate classical observables are mapped consistently to self-adjoint operators in a Hilbert space.

Extending previous work [3], we analyze here the quantum vortex configurations permitted in three-dimensional space. Our main result is that two-dimensional objects such as ribbons and tubes of vorticity have quantum descriptions (subject to the construction of appropriate measures), but one-dimensional vortex filaments do not. Such ribbons or tubes can close on themselves or extend to infinity. In addition to the possibility of their being knotted, they can have global, topological structures such as twists and holes—which lead to quantum properties that cannot occur for filaments. We thus suggest that fundamental, extended quantum configurations are likely to be tubes or ribbons rather than one-dimensional strings, a conjecture with physically important implications.

Our results follow from the construction of representations of an infinite-dimensional algebra and group associated with the dynamical variables of the fluid, utilizing the method of geometric quantization on coadjoint orbits [4]. To obtain a correct kinematic description of quantum vortex configurations associated with the classical Euler equations, it is necessary to incorporate an infinity of degrees of freedom into the quantum theory. When such systems are approximated quasiclassically, or modeled using only finitely many degrees of freedom, qualitatively different conclusions are reached—which is why our findings differ from those of others [5].

Quantization begins with the classical configuration space. For an ideal, incompressible fluid in $\mathbb{R}^3$, a classical configuration is given by a *volume-preserving diffeomorphism of* $\mathbb{R}^3$, i.e., a smooth (C^∞), smoothly invertible mapping ϕ from $\mathbb{R}^3$ to itself whose Jacobian is 1. We can think of $\phi(\mathbf{x})$ as giving the position of a fluid element which, in a fixed reference configuration, is at $\mathbf{x}$. We restrict ourselves to diffeomorphisms describing a fluid stationary at infinity, i.e., $\phi(\mathbf{x})\to\mathbf{x}$ as $|\mathbf{x}|\to\infty$. The set of all such diffeomorphisms is an infinite-dimensional manifold, and forms a group $G=\mathrm{sDiff}(\mathbb{R}^3)$ under the operation of composition (the prefix "s" stands for "special," reflecting the incompressibility condition). Since the physics of the fluid is independent of the choice of coordinates, and the elements of G are coordinate transformations, G acts on itself (the configuration-space manifold) as a *symmetry group.* Thus, as with finite-dimensional symmetry groups, unitarily inequivalent representations of G describe distinct quantum systems associated with such a fluid.

The Lie algebra g of G is the set of smooth, divergenceless vector fields $\mathbf{v}$ on $\mathbb{R}^3$ vanishing at infinity, with the usual Lie bracket (for $\mathbf{v}_1,\mathbf{v}_2\in g$) $[\mathbf{v}_1,\mathbf{v}_2]=(\mathbf{v}_1\cdot\nabla)\mathbf{v}_2-(\mathbf{v}_2\cdot\nabla)\mathbf{v}_1$. Here $\mathbf{v}$ has the interpretation of a *velocity* field. For the algebra to close under the bracket operation, the velocity fields must be C^∞, and are thus only an idealized subset of the possible velocity distributions; they do not describe discontinuities in the fluid's motion. Elements of g belong to the *tangent space* of G at the identity. *Momentum density* fields enter as elements of the *cotangent space* of G; they belong to the dual space g' of g. An element of g' is a continuous, linear mapping from g to $\mathbb{R}$—i.e., a *generalized* vector field in $\mathbb{R}^3$. For $\mathbf{v}\in g$ and $\mathbf{A}\in g'$, we use the notation $\langle\mathbf{A},\mathbf{v}\rangle=\int\mathbf{A}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d^3x$ to express the value of $\mathbf{A}$ on the element $\mathbf{v}$. Now one can *enlarge* the set of physically possible velocity fields to include those proportional to the momentum densities, which embody discontinuities or singularities. Thus, this framework for classical hydrodynamics [2] allows one to begin with smooth configurations, and obtain more singular elements of the phase space *via* the cotangent bundle.

The original velocity fields have the status of *test functions*, with respect to which the more singular momentum density fields are evaluated. Incorporation of a larger set of velocities in this way has no analog for systems with finitely many degrees of freedom, where the tangent space and its dual are finite dimensional and isomorphic.

In the quantized theory, each $\mathbf{v} \in g$ is to be represented by a self-adjoint operator $J(\mathbf{v})$ in a Hilbert space, which is interpreted as the quantum-mechanical momentum density $\mathbf{J}(\mathbf{x})$ *averaged* with the original, smooth velocity field. Such a representation exponentiates to a corresponding unitary representation of G. Representations of diffeomorphism groups and algebras of vector fields have been previously proposed as providing a fundamental unification and classification of quantum systems [6], leading in two dimensions to anyon statistics [7]. In the present context, diffeomorphism group representations offer not only a description of intermediate statistics for vortex configurations, but also a rigorous conclusion about the types of configurations that are quantum mechanically permitted.

We follow the method of geometric quantization to obtain representations of the group and the algebra. First, one characterizes satisfactory *orbits in the coadjoint representation* of G. The coadjoint orbit serves as a *reduced classical phase space* [2]. When the Hamiltonian respects the group symmetry, the phase-space trajectory of the initial-value problem remains in the orbit, constraining the classical motion to particular values for (possibly infinitely many) conserved quantities. To be suitable for quantization, a coadjoint orbit must have several properties. It must admit a *polarization*, which means that the dynamical variables can be divided into two complementary sets, where information about one set is lost when measurements are made on the other set. The existence of a polarization thus expresses the uncertainty principle. It must also obey an *integrality condition* which ensures that an appropriate periodic boundary condition is possible, defining domains of wave functions in Hilbert space for the operator observables in the resulting quantum theory. Finally, a suitable orbit (or union of uncountably many orbits) must carry a *measure*, quasi-invariant under the diffeomorphism group action. This means that it is possible to calculate the expected values of physical observables with respect to underlying probabilities. On such orbits, one can construct *continuous, unitary, irreducible representations* of G, for which all the desired self-adjoint operators can be recovered as generators of unitary subgroups [6]. Elements of an orbit correspond to phase-space configurations; a *foliation* of the orbit (given by the polarization) yields the quantum configuration space [4].

We next construct a polarization for coadjoint orbits of $G = \mathrm{sDiff}(\mathsf{R}^3)$ containing ribbon and tube configurations of vorticity, and explain why none exists for orbits containing one-dimensional filament configurations. This extends our earlier result [3] that in R^2 polarizations exist for orbits containing filaments of vorticity, but not for pure point vortices.

Since $\mathbf{v}$ is divergenceless, $\mathbf{v} = \nabla \times \boldsymbol{\chi}$, where $\boldsymbol{\chi}$ is a (covariant) vector-valued stream function on R^3, defined up to addition of an arbitrary gradient. We write $\boldsymbol{\chi}_{\mathbf{v}}$ when we want to emphasize the dependence of $\boldsymbol{\chi}$ on $\mathbf{v}$. Now

$$\begin{aligned}\int \mathbf{A}(\mathbf{x})\cdot\mathbf{v}(\mathbf{x})d^3x &= \int \mathbf{A}(\mathbf{x})\cdot(\nabla\times\boldsymbol{\chi})(\mathbf{x})d^3x \\ &= -\int\!\!\int(\mathbf{A}\times\boldsymbol{\chi})\cdot\mathbf{n}\,d^2s \\ &\quad + \int(\nabla\times\mathbf{A})(\mathbf{x})\cdot\boldsymbol{\chi}(\mathbf{x})d^3x\,,\end{aligned}$$

where the area integral is taken around the surface of a closed sphere of very large diameter. Because $\mathbf{v} \to 0$ rapidly as $|\mathbf{x}| \to \infty$, we can choose $\boldsymbol{\chi}$ so that $\boldsymbol{\chi} \to 0$ rapidly as $|\mathbf{x}| \to \infty$. This allows us to drop the surface term, writing $\langle \mathbf{A},\mathbf{v}\rangle = \int \boldsymbol{\Omega}(\mathbf{x})\cdot\boldsymbol{\chi}(\mathbf{x})d^3x = \langle \boldsymbol{\Omega},\boldsymbol{\chi}\rangle$. We see that the value of $\mathbf{A}$ as an element of g' depends only on its curl, the (contravariant) vorticity density Ω (having set $m=1$). Note that the surface term vanishes as long as $\boldsymbol{\chi} \to 0$ sufficiently rapidly, even when $\mathbf{A}$ itself does not vanish at infinity. It is useful to note that $\nabla \times (\mathbf{v}_1 \times \mathbf{v}_2) = [\mathbf{v}_1,\mathbf{v}_2]$, so that one choice is $\boldsymbol{\chi}_{[\mathbf{v}_1,\mathbf{v}_2]} = \mathbf{v}_1 \times \mathbf{v}_2$.

The adjoint representation $\mathrm{Ad}\mathbf{v}$ of g is defined so that for all $\mathbf{v}_1,\mathbf{v}_2 \in g$, $(\mathrm{Ad}\mathbf{v}_1)\mathbf{v}_2 = [\mathbf{v}_1,\mathbf{v}_2]$. Then the adjoint representation of G on g is given, for $\phi \in G$, by $\mathbf{v}' = (\mathrm{Ad}\phi)\mathbf{v} = [J_{\phi^{-1}}\mathbf{v}]\circ\phi$, where J denotes the matrix of derivatives; i.e., $J_\phi = [J_\phi]^j_k = \partial_k\phi^j$. Correspondingly, the action of ϕ on $\boldsymbol{\chi}$ is given by $\boldsymbol{\chi}' = (\mathrm{Ad}\phi)\boldsymbol{\chi} = [J_\phi]^T[\boldsymbol{\chi}\circ\phi]$, where the superscript T denotes the transpose. The coadjoint representation specifies the action of G in the dual space g'. For $\mathbf{A} \in g'$, we write $\mathbf{A}' = \phi\mathbf{A}$, where $\langle \mathbf{A}',\mathbf{v}\rangle = \langle \mathbf{A},\mathrm{Ad}(\phi^{-1})\mathbf{v}\rangle$. Then $\mathbf{A}' = [J_\phi]^T[\mathbf{A}\circ\phi]$. Finally, the coadjoint representation may also be written in terms of the action of G on the vorticity $\boldsymbol{\Omega}$; one obtains $\boldsymbol{\Omega}' = ([J_{\phi^{-1}}]\boldsymbol{\Omega})\circ\phi$. A coadjoint orbit is thus obtained by considering a particular vorticity distribution $\boldsymbol{\Omega}(\mathbf{x})$, and acting on it with all the diffeomorphisms in $\mathrm{sDiff}(\mathsf{R}^3)$.

We first consider an infinite ribbon of vorticity, on a surface bounded by two lines and extending to infinity in both z directions. This element of g' is given by the surface Σ in R^3, together with a (singular) vorticity density distribution $\boldsymbol{\gamma}(\mathbf{s})$ tangent to Σ (for $\mathbf{s} \in \Sigma$) and tangential on the boundary of Σ. That is, $\boldsymbol{\Omega}$ is defined by $\langle \boldsymbol{\Omega},\boldsymbol{\chi}\rangle = \int_\Sigma \boldsymbol{\gamma}(\mathbf{s})\cdot\boldsymbol{\chi}(\mathbf{s})d^2s$; and we identify $\boldsymbol{\Omega}$ with the pair $(\Sigma,\boldsymbol{\gamma})$. The ribbon then consists of a family of one-dimensional "lines of vorticity." Let Γ be any smooth curve crossing the ribbon in a transverse direction, from one bounding line to the other. The total vorticity of the ribbon is given by $\Omega_{\mathrm{tot}} = \int_\Gamma d\mathbf{l}\cdot(\mathbf{n}\times\boldsymbol{\gamma})$, where $\mathbf{n}$ is the unit normal to Σ on Γ; Ω_{tot} is independent of Γ (and thus a constant along the ribbon). In the coadjoint representation of $\mathrm{sDiff}(\mathsf{R}^3)$, the action of a diffeomorphism ϕ on $\boldsymbol{\Omega} = (\Sigma,\boldsymbol{\gamma})$ is to give a transformed surface and a new vorticity density tangent to the surface: Writing $(\Sigma',\gamma')=\phi(\Sigma,\gamma)$, we obtain $\Sigma'=\{\mathbf{s}'=\phi^{-1}(\mathbf{s})|\mathbf{s}\in\Sigma\}$, and $\gamma'(\mathbf{s}')=(\mathbf{n}'(\mathbf{s}')\cdot\{[J_{\phi^{-1}}\mathbf{n}]\circ\phi\}(\mathbf{s}'))\{([J_{\phi^{-1}}]\gamma)\circ\phi\}(\mathbf{s}')$, where $\mathbf{n}'$ is the unit normal to Σ' at $\mathbf{s}'$ (the first factor here is just the Jacobian of the area element d^2s with respect to d^2s'). The set of such vortex ribbons (fixed at infinity) forms a coadjoint orbit for each value of Ω_{tot}.

For specificity consider the special case $\Omega_0=(\Sigma_0,\gamma_0)$, where $\Sigma_0=\{(x,y,z)\in\mathrm{R}^3|0<x<1,\ y=0\}$ and $\gamma_0(\mathbf{s})=\gamma_0\hat{\mathbf{z}}$. The stability group K_{Ω_0} of Ω_0 consists of those diffeomorphisms which not only preserve Σ_0 as a set, but also preserve individual lines of vorticity within Σ_0; i.e., with $(x',y',z')=\phi(x,y,z)$, we have $\phi\in K_{\Omega_0}$ if for $(x,y,z)\in\Sigma_0$, $x'=x$ and $y'=0$. The Lie algebra k_{Ω_0} of K_{Ω_0} consists of vector fields which on Σ_0 are not only tangential to Σ_0, but are in the direction of γ_0. To obtain a polarization, we now *relax* the constraint that the diffeomorphisms preserve the lines of vorticity, so that the only restriction on ϕ is that it preserve Σ_0 as a set. Letting H_{Ω_0} be this subgroup of $\mathrm{sDiff}(\mathrm{R}^3)$, the corresponding Lie algebra h_{Ω_0} consists of vector fields which are tangential to Σ_0 (on Σ_0), as well as tangential to its boundary lines $x=y=0$ and $x=1$, $y=0$. To check that this is a polarization, one must verify the condition that $\langle\Omega_0,\chi_{[\mathbf{v}_1,\mathbf{v}_2]}\rangle=0$ for all $\mathbf{v}_1,\mathbf{v}_2\in h_{\Omega_0}$. But this holds because $\chi_{[\mathbf{v}_1,\mathbf{v}_2]}=\mathbf{v}_1\times\mathbf{v}_2$, which on Σ_0 is orthogonal to Σ_0 (and hence to γ_0). Now we can define a corresponding character ω on H_{Ω_0}. Let $\phi_t^{\mathbf{v}}$ (for $t\in\mathrm{R}$) be the one-parameter group of volume-preserving diffeomorphisms generated by a velocity field $\mathbf{v}\in h_{\Omega_0}$; ω must satisfy

$$(d/dt)\ln[\omega(\phi_t^{\mathbf{v}})]=i\langle\Omega_0,\chi_{\mathbf{v}}\rangle\,.$$

Let Γ_0 be any line of vorticity in Σ_0, and let Γ_∞ be a line at infinity parallel to the z axis and not in the plane of the ribbon. Consider a surface S_0 bounded by Γ_0 and Γ_∞. Let $V(t)$ be the signed volume of the region swept out by $\phi_{t'}^{\mathbf{v}}S_0$ as t' varies from 0 to t and the surface moves from S_0 to $\phi_t^{\mathbf{v}}S_0$. The volume is finite because $\phi(\mathbf{x})\to\mathbf{x}$ rapidly at infinity. Furthermore, $V(t)$ is independent of Γ_0, Γ_∞, and S_0 because ϕ is a volume-preserving flow. Finally, from Stokes's integral formula, we obtain $\gamma_0(d/dt)V(t)=\langle\Omega_0,\chi_{\mathbf{v}}\rangle$. Then the desired character satisfies $\omega(\phi_t^{\mathbf{v}})=\exp[i\gamma_0V(t)]$. Because the geometric construction of $V(t)$ generalizes to any $\phi\in H_{\Omega_0}$, ω extends (as a continuous group homomorphism) from the one-parameter flows to all of H_{Ω_0}. The explicit construction of a character means that the integrality condition holds for this coadjoint orbit of G (but see the discussion below of quantization of vorticity). Thus, for ribbons of vorticity, we have completed all but the difficult step of obtaining a quasi-invariant measure for the group action —which is beyond the scope of the present Letter.

Having obtained H_{Ω_0}, the configuration space Δ is the quotient space G/H_{Ω_0}; its elements are the ribbons Σ with total vorticity Ω_{tot}, but without the information as to how the vorticity is distributed. The uncertainty principle requires that γ cannot be measured simultaneously with Σ. The character ω of H_{Ω_0} now induces a unitary representation of G, leading to the self-adjoint operators $J(\mathbf{v})$ acting in the Hilbert space of square-integrable functions on Δ.

Parallel arguments apply when the side edges of the ribbon are identified, when the ribbon joins with itself rather than extending to infinity, or when the ribbon separates into two, rejoins, and so on. Thus a polarization also exists for vortex tubes, toruses, ribbons with holes, etc.

The integrality condition is automatically satisfied in the above example, because our diffeomorphisms become trivial at infinity. Rotation of the infinite ribbon by 2π leaves a "twist" in the diffeomorphism implementing the rotation, so that the requirement of integrality imposes no new constraint. But if g is taken to include generators of global rotations, integrality becomes nontrivial. For the infinite vortex ribbon in a large but finite volume $\mathcal{V}$, the integrality condition takes the form $\mathcal{V}\Omega_{\text{tot}}=2\pi N$, where N is an integer. (In the preceding we avoided well-known divergences proportional to the volume by working with local densities as the dynamical variables.) This is equivalent to the Feynman-Onsager condition [8] when $\kappa=\oint\mathbf{v}\cdot d\mathbf{l}=(\mathcal{V}/m)\Omega_{\text{tot}}$; in the proper physical units, with $[J(\mathbf{v}_1),J(\mathbf{v}_2)]=-i\hbar J([\mathbf{v}_1,\mathbf{v}_2])$, this results in the quantization of vorticity $\kappa=(h/m)N$.

Two-dimensional distributions of vorticity embedded in three dimensions have field degrees of freedom distinct from one-dimensional filaments. There are also discrete degrees of freedom associated with the topology of these vortex configurations: For instance, a closed ribbon and a twisted closed ribbon of vorticity belong to different coadjoint orbits, and lead to unitarily inequivalent representations of $\mathrm{sDiff}(\mathrm{R}^3)$. Such configurations can be distinguished by assigning to them an integer counting the number of twists. The additional degrees of freedom should in principle lead to observable effects—for example, in the value of the specific heat a fluid whose excitations include twisted and untwisted ribbons of vorticity.

We next show that a coadjoint orbit consisting of pure (one-dimensional) vortex filaments in three-dimensional space has no polarization. Such a filament is uniquely characterized by an unparametrized curve Γ and the magnitude of the (singular) vorticity density γ with support on Γ, and is defined by $(\Omega_\Gamma,\chi_{\mathbf{v}})=\gamma\int_\Gamma d\mathbf{s}\cdot\chi_{\mathbf{v}}$. The algebra of the little group k_Γ consists of the divergenceless vector fields which on Γ are tangent to Γ. The polarization algebra h_Γ must be a proper subalgebra of g which contains k_Γ, and satisfy the condition $(\Gamma,\chi_{[\mathbf{h}_1,\mathbf{h}_2]})=0$ for $\mathbf{h}_1,\mathbf{h}_2\in h_\Gamma$. This implies that $\int_\Gamma d\mathbf{s}\cdot(\mathbf{h}_1\times\mathbf{h}_2)=0$. The latter condition *can* be satisfied—for example, by letting h_Γ contain vector fields which at each point $\mathbf{s}\in\Gamma$ take values in a *plane* formed by Γ and a single direction perpendicular to Γ at $\mathbf{s}$. The cross product of two such vectors remains perpendicular to Γ. However, such vector fields cannot form a closed Lie subalgebra. To see this choose any $\mathbf{v} \in h_\Gamma$ with $h \notin k_\Gamma$. There must be a point $p \in \Gamma$ such that $\mathbf{v}(p)$ is not tangent to Γ. Because Γ may be transformed by an element of G, we may assume without loss of generality that near p the filament is a straight line in the z direction, and that $\mathbf{v}$ lies in the x-z plane. Now for $\mathbf{v}' \in k_\Gamma$ consider (at the point p) $\hat{\mathbf{y}} \cdot [\mathbf{v},\mathbf{v}'] = v_x \partial_x v'_y$. But there is no restriction on the derivatives of an element of k_Γ; thus this quantity may be nonzero, and $[\mathbf{v},\mathbf{v}']$ will in general have a component in the y direction at p. Thus the bracket leads to arbitrary elements of g. Once one element of the polarization group moves a point off the filament, composition with the little group (which is not restricted off the filament) leads to any group element. Thus polarizations for orbits of vortex filaments do not exist. Introduction of a complex structure on the coadjoint orbit cannot overcome this fundamental difficulty. Penna and Spera [8], working in the same framework as the present article, studied vortex filament orbits for sDiff(R^3). However, the results reported here show that such orbits are not suitable for quantization.

In the present framework, particular values for the conserved quantities reduce the phase space to a coadjoint orbit. When a polarization exists and integrality holds, a representation of the full algebra is obtained. Alternatively, one can attempt to work with fewer coordinates from the outset [5]. For example, considering only special linear transformations and translations of R^2, one obtains a model studied recently by Leinaas and Myrheim [7], which accommodates point vortices. Classically, sDiff(R^2) also describes point vortices. However, the approaches lead to fundamentally different results on the quantum level—quantizing the finite-dimensional theory allows quantum point vortices, but quantizing the full theory does not. Thus qualitatively different results emerge when an infinite number of degrees of freedom are included. We also mention a result of Shishkov [5] in which it was shown that an approximate description of point vortices can be obtained by choosing the velocity fields to belong to a certain orthonormal family. However, the algebra of vector fields is satisfied only to first order, and there is no representation of the diffeomorphism group to which this is an approximation. Thus, for the purpose of identifying which kinds of vortex configurations are possible in a quantum theory, a finite-dimensional model or a quasiclassical approximate theory may not be adequate.

In closing, we observe that establishing which orbits are quantizable is not sufficient to ensure correct quantization. A very careful treatment of measures (or half-forms) on the orbits is also required. We stress, however, that surfaces of vorticity as described here are necessary; the impossibility of quantizing the vortex filament is a rigorous result.

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