Gauge Theory of the Virasoro Group

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A gauge theory of the Virasoro group is constructed as a theory of "internal" strings, and a spontaneous symmetry-breaking mechanism of the Virasoro group down to its Cartan subgroup is discussed. It is shown that the mass spectrum of the gauge bosons after the symmetry breaking becomes that of a harmonic oscillator. After a supersymmetric generalization, the theory could serve as a simple model for the hadrons.

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It has long been recognized that the Virasoro group [1] is *the* fundamental symmetry of the string theory [2]. This implies that any realistic four-dimensional field-theoretic description of the string theory (before an inevitable symmetry breaking) is most likely to be formulated as a gauge theory of the Virasoro group. So far, however, such a gauge theory has not been available. The reason is partly due to our lack of understanding of the representation of the infinite-dimensional group. The purpose of this Letter is to construct a genuine gauge theory of the Virasoro group which could serve as a realistic model for the hadrons, and to discuss a spontaneous symmetry-breaking mechanism of the theory. Remarkably, our result shows that *the mass spectrum of the gauge bosons after the symmetry breaking becomes that of a harmonic oscillator.* Specifically, the mass of the Hermitian gauge fields $A_\mu{}^k$ (k an integer) is shown to be $g\omega|k|/2$, where g is the coupling constant and ω is the symmetry-breaking mass scale, which should be related to the string tension in the real world.

Before we construct the desired gauge theory, however, we need to know some basic facts about the representation of the Virasoro algebra. The representation most widely discussed in the literature is the highest-weight representation [3] based on the Verma module. Unfortunately this representation is not very useful in constructing a gauge theory of the Virasoro group because the gauge field (which must form an adjoint representation) does not belong to this representation. The representation that we use in this Letter is a generalization of the one introduced by Kaplansky [4], and also independently by Feigin and Fuks [5]. To explain the generalization of the Kaplansky-Feigin-Fuks (KFF) representation, we first consider the Virasoro algebra without the central extension,

$$[\xi_m,\xi_n]=f_{mn}{}^k\xi_k =(m-n)\delta_{m+n}{}^k\xi_k \quad (k,m,n \text{ integers})\,. \tag{1}$$

From the commutation relation we automatically obtain *the adjoint representation* L_m,

$$(L_m)_n{}^k=f_{mn}{}^k=(m-n)\delta_{m+n}{}^k\,, \tag{2}$$

which acts on an infinite-dimensional vector ϕ^k as

$$(L_m\phi)^k=-f_{mn}{}^k\phi^n=-(2m-k)\phi^{k-m}\,. \tag{3}$$

The KFF representation [4,5] may be understood as a natural extension of the adjoint representation, and is given by the following two-parameter family of the infinite-dimensional matrix representation $L_m^{(\alpha,\beta)}$ (or L_m for short) which we call *the* (α,β) *representation*:

$$(L_m)_n{}^k=[(\alpha+1)m+\beta-k]\delta_{m+n}{}^k\,, \tag{4}$$

where α and β are arbitrary complex numbers which characterize the representation. The representation (4) acts on an infinite-dimensional vector space which we call the (α,β) module $V_{(\alpha,\beta)}$ (or V for short) as follows [4,6]:

$$(L_m\phi)^k=-(L_m)_n{}^k\phi^n =-[(\alpha+1)m+\beta-k]\phi^{k-m}\,, \tag{5}$$

where ϕ^k is an element of V. Notice that indeed the adjoint representation (2) is nothing but a particular case of the KFF representation, i.e., the (1,0) representation.

The KFF representation allows us to introduce *the dual representation* which acts on the vector space V^* dual to V as follows:

$$(L_m\omega)_k=(L_m)_k{}^n\omega_n=(\alpha m+\beta-k)\omega_{k+m}\,, \tag{6}$$

where ω_k is an element of V^*. With the introduction of the dual module V^* we can now define *the contraction*, or the scalar product, of the two vectors ϕ^k and ω_k which is invariant under the Virasoro transformation [6],

$$L_m(\phi^k\omega_k)=0\,. \tag{7}$$

In fact, the dual representation is defined precisely to allow such an invariant scalar product.

Now we generalize the KFF representation to a tensor module of (p,q) type, $T_{p,q}$. Let $t_{i\cdots j}{}^{k\cdots l}$ be an infinite-dimensional (p,q) tensor which has p upper indices and q lower ones, which forms an element of $T_{p,q}$. We define *the tensor representation* $\hat{L}_m$ by [6]

$$(\hat{L}_mt)_{i\cdots j}{}^{k\cdots l}=(L_m)_i{}^nt_{n\cdots j}{}^{k\cdots l}+\cdots+(L_m)_j{}^nt_{i\cdots n}{}^{k\cdots l}-(L_m)_n{}^kt_{i\cdots j}{}^{n\cdots l}-\cdots-(L_m)_n{}^lt_{i\cdots j}{}^{k\cdots n}\,. \tag{8}$$

From the definition it becomes clear that $\hat{L}_m$ forms a representation of the Virasoro algebra. Indeed it is the tensor product of the (α,β) representation. This shows that the KFF representation can easily be generalized to an arbitrary tensor module of (p,q) type. In this generalization ϕ^k and ω_k become a (1,0) tensor and a (0,1) tensor, respectively. One may proceed further to define *a tensor representation of mixed type*, by requiring each of the $p+q$ indices in (8) to transform according to different (α,β) representations. For example, one can define a (0,2) tensor of mixed type $\hat{t}_{ij}$, where i belongs to an (α,β) representation but j belongs to an (α',β') representation. By definition we have

$$(\hat{L}_m\hat{t})_{ij}=(\alpha m+\beta-i)\hat{t}_{i+m,j}+(\alpha' m+\beta'-j)\hat{t}_{i,j+m}\,, \tag{9}$$

from which it follows that

$$([\hat{L}_m,\hat{L}_n]\hat{t})_{ij}=(m-n)(\hat{L}_{m+n}\hat{t})_{ij}\,. \tag{10}$$

This shows that one can indeed have a (p,q) tensor of mixed type. With the above generalization we observe the following [6].

(1) The Kronecker $\delta_i{}^j$ is an invariant tensor of any (α,β) representation.

(2) The metric $g_{ij}=\delta_{i+j}{}^0$ is an invariant tensor of the $(-\frac{1}{2},0)$ representation. The existence of the invariant metric makes the $(-\frac{1}{2},0)$ representation very important in physical applications.

(3) The tensor $\hat{f}_{ij}{}^k$ defined by

$$\hat{f}_{ij}{}^k=(L_i)_j{}^k=[(\alpha+1)i+\beta-k]\delta_{i+j}{}^k \tag{11}$$

is an invariant tensor of mixed type, where i,j,k belong to (1,0), (α,β), and (α,β) representations, respectively. This follows from

$$\begin{aligned}(\hat{L}_m\hat{f})_{ij}{}^k=&(m-i)\hat{f}_{i+m,j}{}^k+(\alpha m+\beta-j)\hat{f}_{i,j+m}{}^k\\&-[(\alpha+1)m+\beta-k]\hat{f}_{ij}{}^{k-m}\\=&0\,.\end{aligned}$$

(4) The tensor $\hat{d}_{ij}{}^k$ defined by

$$\hat{d}_{ij}{}^k=\delta_{i+j}{}^k \tag{12}$$

is an invariant tensor of mixed type, where i,j,k belong to (α_1,β_1), (α_2,β_2), and $(\alpha_1+\alpha_2,\beta_1+\beta_2)$ representations, respectively. Similarly, the tensor $\hat{d}_k{}^{ij}$ defined by

$$\hat{d}_i{}^{jk}=\delta_i{}^{j+k} \tag{13}$$

is an invariant tensor of mixed type, where i,j,k belong to (α_1,β_1), (α_2,β_2), and $(\alpha_1-\alpha_2-1,\beta_1-\beta_2)$ representations, respectively.

The existence of the invariant tensors $\hat{f}_{ij}{}^k$, $\hat{d}_{ij}{}^k$, and $\hat{d}_i{}^{jk}$ allows us to define *the vector products*, the $\hat{f}$ product (the cross product) and the $\hat{d}$ product (the symmetric product), between two vectors to obtain another. Given an adjoint representation θ^i and an (α,β) representation ϕ^i, one can define the $\hat{f}$ product $(\theta\times\phi)^k$ by

$$(\theta\times\phi)^k=\hat{f}_{ij}{}^k\theta^i\phi^j=[(\alpha+1)m+\beta-k]\theta^m\phi^{k-m}\,. \tag{14}$$

Clearly the $\hat{f}$ product gives us an element of an (α,β) module,

$$\begin{aligned}[L_m(\theta\times\phi)]^k&=[(L_m\theta)\times\phi]^k+[\theta\times(L_m\phi)]^k\\&=-[(\alpha+1)m+\beta-k](\theta\times\phi)^{k-m}\,.\end{aligned} \tag{15}$$

By the same reasoning, given an (α_1,β_1) representation ϕ_1^k and an (α_2,β_2) representation ϕ_2^k, one can define the $\hat{d}$ product $(\phi_1 * \phi_2)^k$ by

$$(\phi_1*\phi_2)^k=\hat{d}_{ij}{}^k\phi_1^i\phi_2^j=\phi_1^m\phi_2^{k-m} \tag{16}$$

to obtain an element of an $(\alpha_1+\alpha_2,\beta_1+\beta_2)$ module. The $\hat{d}$ product allows us to define the "square" of an arbitrary (α,β) representation ϕ^k,

$$(\phi^2)^k=(\phi*\phi)^k=\phi^m\phi^{k-m}\,,$$

or in general the "nth power" $(\phi^n)^k$ by

$$(\phi^n)^k=\sum_{k_1+k_2+\cdots+k_n=k}\phi^{k_1}\phi^{k_2}\cdots\phi^{k_n} \tag{17}$$

to obtain an element of an $(n\alpha,n\beta)$ module. Similarly, one can define the nth power of an (α,β) representation ω_k by

$$(\omega^n)_k=\sum_{k_1+k_2+\cdots+k_n=k}\omega_{k_1}\omega_{k_2}\cdots\omega_{k_n} \tag{18}$$

to obtain an element of an $(n\alpha+n-1,n\beta)$ module. Of course the vector products can also be defined between vectors of different types. For example, given two (α,β) representations ϕ^k and ω_k, one can define the $\hat{f}$ product $(\phi\times\omega)_k$ by

$$\begin{aligned}(\phi\times\omega)_k&=\hat{f}_{ki}{}^j\phi^i\omega_j\\&=(\alpha k+\beta-m)\phi^m\omega_{k+m}\end{aligned} \tag{19}$$

to obtain an element of an adjoint module.

With the above preliminaries we are ready to discuss a gauge theory of the Virasoro group. For this we first introduce the gauge potential $A_\mu{}^k$ which forms an adjoint representation. As for the matter field we consider for simplicity a scalar multiplet ϕ^k or ω_k which forms an (α,β) representation. Now let θ^k be the infinitesimal gauge parameter of the Virasoro group which forms an adjoint representation. Under an infinitesimal gauge transformation we require

$$\begin{aligned}\delta A_\mu{}^k&=-(1/g)[\partial_\mu\theta^k+ig(A_\mu\times\theta)^k]\\&=-(1/g)[\partial_\mu\theta^k+ig(2m-k)A_\mu{}^m\theta^{k-m}]\,,\\\delta\phi^k&=i(\theta\times\phi)^k\\&=i[(\alpha+1)m+\beta-k]\theta^m\phi^{k-m}\,,\\\delta\omega_k&=-i(\theta\times\omega)_k\\&=-i(\alpha m+\beta-k)\theta^m\omega_{k+m}\,,\end{aligned} \tag{20}$$

where g is the coupling constant. Next, we define the field strength $F_{\mu\nu}{}^k$ by

$$F_{\mu\nu}{}^k = \partial_\mu A_\nu{}^k - \partial_\nu A_\mu{}^k + ig(A_\mu \times A_\nu)^k$$
$$= \partial_\mu A_\nu{}^k - \partial_\nu A_\mu{}^k + ig(2m-k) A_\mu{}^m A_\nu{}^{k-m}\,, \qquad (21)$$

and the covariant derivative D_μ by

$$D_\mu \phi^k = \partial_\mu \phi^k + ig(A_\mu \times \phi)^k$$
$$= \partial_\mu \phi^k + ig[(\alpha+1)m + \beta - k] A_\mu{}^m \phi^{k-m}\,,$$
$$D_\mu \omega_k = \partial_\mu \omega_k - ig(A_\mu \times \omega)_k$$
$$= \partial_\mu \omega_k - ig(\alpha m + \beta - k) A_\mu{}^m \omega_{k+m}\,. \qquad (22)$$

Notice that our definition of the field strength (21) is different from the conventional one in that the self-interaction couplings as well as the minimal coupling to the matter of the gauge potential is characterized by ig, not g. With the definitions we observe the following [6].

(1) The field strength $F_{\mu\nu}{}^k$ transforms as an adjoint representation under the gauge transformation,

$$\delta F_{\mu\nu}{}^k = i(\theta \times F_{\mu\nu})^k\,. \qquad (23)$$

(2) The covariant derivatives transform covariantly under the gauge transformation,

$$\delta(D_\mu \phi^k) = i(\theta \times D_\mu \phi)^k\,,$$
$$\delta(D_\mu \omega_k) = -i(\theta \times D_\mu \omega)_k\,. \qquad (24)$$

Although all the above equalities hold with a complex g, we will require g to be real in this Letter for a reason which will become clear soon.

Before we introduce the Lagrangian, we need the concept of Hermiticity. We call a tensor field $t_{i\cdots j}{}^{k\cdots l}$ *Hermitian* if

$$(t_{i\cdots j}{}^{k\cdots l})^* = t_{-i\cdots -j}{}^{-k\cdots -l}\,. \qquad (25)$$

Notice that, when g is real, one has for Hermitian fields

$$(D_\mu \phi^k)^* = D_\mu \phi^{-k}, \quad (D_\mu \omega_k)^* = D_\mu \omega_{-k}\,, \qquad (26)$$

if α is real and β is imaginary. For this reason we will call an (α,β) representation *Hermitian* if α is real and β is imaginary. With this we notice the following [6].

(1) When g is real, the field strength of a Hermitian potential $A_\mu{}^k$ becomes Hermitian,

$$(F_{\mu\nu}{}^k)^* = F_{\mu\nu}{}^{-k}\,. \qquad (27)$$

(2) The $\hat{f}$ product of two Hermitian fields defined by a Hermitian representation is *anti-Hermitian.* For example, we have

$$[(\theta\times\phi)^k]^* = [(\alpha+1)m + \beta - k]^* \theta^{-m} \phi^{-k+m}$$
$$= -(\theta\times\phi)^{-k}\,. \qquad (28)$$

(3) The $\hat{d}$ product of two Hermitian fields is Hermitian.

Now we have all the necessary ingredients to construct a gauge theory of the Virasoro group. Let $A_\mu{}^k$ be the Hermitian gauge potential, ω_k be a Hermitian field which forms a $(-\frac{1}{2},0)$ representation, and consider the following Lagrangian:

$$\mathcal{L} = -\tfrac{1}{4}\kappa^6 \hat{d}_{ij}{}^k (\omega^6)_k F_{\mu\nu}{}^i F_{\mu\nu}{}^j + \tfrac{1}{2}(D_\mu \omega_{-k})(D_\mu \omega_k) + \tfrac{1}{2}\mu^2 \omega_{-k}\omega_k - \tfrac{1}{4}\lambda(\omega_{-k}\omega_k)^2$$
$$= -\tfrac{1}{4}\kappa^6 (\omega^6)_{i+j} F_{\mu\nu}{}^i F_{\mu\nu}{}^j + \tfrac{1}{2}(D_\mu \omega_k)^*(D_\mu \omega_k) + \tfrac{1}{2}\mu^2 \omega_k^* \omega_k - \tfrac{1}{4}\lambda(\omega_k^*\omega_k)^2\,, \qquad (29)$$

where we have introduced a scale parameter κ to keep $k\omega_k$ dimensionless. Notice that the Lagrangian is explicitly real. Furthermore, it is manifestly invariant under the gauge transformation (20), which is made possible due to the existence of the invariant tensor $\hat{d}_{ij}{}^k$ and the invariant metric $\delta_{i+j}{}^0$ of the $(-\frac{1}{2},0)$ representation. To check the gauge invariance notice that, since $F_{\mu\nu}{}^i$ forms a (1,0) representation, the $\hat{d}$ product of two field strengths forms a (2,0) representation. On the other hand, $(\omega^6)_k$ also forms a (2,0) representation due to (18). So the first term in the Lagrangian is explicitly gauge invariant. The other terms are invariant because they are the scalar products of two $(-\frac{1}{2},0)$ representations obtained with the invariant metric $\delta_{i+j}{}^0$.

Clearly the Lagrangian describes a genuine gauge theory of the Virasoro group. Significantly, it can break the gauge symmetry spontaneously down to the Cartan subgroup U(1), because ω_k can play the role of the Higgs multiplet. To see this let

$$\langle \omega_k \rangle = \omega \delta_k{}^0 = (\mu^2/\lambda)^{1/2} \delta_k{}^0\,, \qquad (30)$$

and obtain

$$\langle (\omega^6)_k \rangle = \omega^6 \delta_k{}^0\,.$$

So with $\kappa = \omega^{-1}$ and with

$$\omega_0 = \omega + \phi\,,$$
$$B_\mu{}^k = A_\mu{}^k + (i/g\omega)(2/k)\partial_\mu \omega^k \quad (k \neq 0)\,,$$

where $\omega^k = \delta_0{}^{k+l}\omega_l = \omega_{-k}$, the Lagrangian (29) can be written as

$$\mathcal{L} = -\tfrac{1}{4} G_{\mu\nu}{}^{-k} G_{\mu\nu}{}^k - \tfrac{1}{2} g^2 \omega^2 (k/2)^2 B_\mu{}^{-k} B_\mu{}^k + \tfrac{1}{2}(\partial_\mu \phi)(\partial_\mu \phi) - \mu^2 \phi^2 + \mu^4/4\lambda + \text{higher-order terms}\,, \qquad (31)$$

where

$$G_{\mu\nu}{}^0 = \partial_\mu A_\nu{}^0 - \partial_\nu A_\mu{}^0, \quad G_{\mu\nu}{}^k = \partial_\mu B_\nu{}^k - \partial_\nu B_\mu{}^k \quad (k \neq 0)\,.$$

From this we find that the gauge field $A_\mu{}^k$ transforms into the massive vector field $B_\mu{}^k$ which has the mass

$$m_k = \frac{1}{2} g\omega|k| = \frac{1}{2} g(\mu^2/\lambda)^{1/2}|k| \quad (k \text{ integer}) \tag{32}$$

after the spontaneous symmetry breaking. Furthermore, the real scalar field ϕ becomes the Higgs field of the symmetry breaking, and acquires a mass $m=\sqrt{2}\mu$. Only $A_\mu{}^0$ remains massless, and all the massive modes except the Higgs field ϕ become doubly degenerate.

At this point one may wonder why a gauge theory of the Virasoro group has not been available so far. The reason is obvious. The Virasoro group is not only noncompact, but also does not admit any bi-invariant Cartan-Killing metric. To make matters worse, the (α,β) representation (in particular the adjoint representation) of the Virasoro group does not form a unitary representation in general [6]. Indeed it becomes unitary only if $\alpha+\alpha^*+1=0$ and $\beta-\beta^*=0$. Under this circumstance one cannot construct a gauge theory of the Virasoro group with any known method. In our approach we show explicitly how to construct a gauge theory of the Virasoro group in the absence of a positive definite bi-invariant metric. Notice that, in spite of the fact that the adjoint representation of the Virasoro group is nonunitary, the unitarity of the theory is guaranteed after the spontaneous symmetry breaking of the Virasoro group down to the Cartan subgroup U(1). Indeed *all the physical fields become explicitly unitary under the U(1) subgroup after the symmetry breaking.* This must be obvious from (31). This guarantees the positive definiteness of the Hamiltonian. Notice that the concept of Hermiticity plays the crucial role to make the Hamiltonian positive definite.

In conclusion, we have constructed a genuine gauge theory of the Virasoro group which can give rise to a spontaneous symmetry breaking. Mathematically the Lagrangian (29) could be interpreted as a bosonic theory of a closed string in which the string degrees of freedom becomes purely internal [7]. But from the practical point of view it could serve as a simple model for the mesons, or more generally as a realistic model for the hadrons after a proper supersymmetric generalization. Just like the string theory our theory has a dimensional parameter κ, which obviously should be related to the hadronic scale characterized by the string tension in the real world. Although the mass spectrum (32) does not fully describe the spectrum of the mesons to which it was intended, certainly the theory has many attractive features. A more detailed discussion of the theory with a Kac-Moody extension and a supersymmetric generalization will be published elsewhere [7].

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