Nonlinear Heat Transport near the Superfluid Transition of ^{4}He

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We investigate the effect of a finite heat current Q on the superfluid transition of ^{4}He. We perform a renormalization-group (RG) calculation of the temperature profile and of the critical thermal conductivity $\lambda_T(Q)$ in the nonlinear-response regime. In the experimentally accessible region close to T_λ we predict the divergence $\lambda_T(Q) \sim Q^{-y_\lambda}$ with the effective exponent $y_\lambda \approx 0.31$. An experiment is proposed to detect this nonlinear effect. We also present the exact RG result $x=(2\nu)^{-1}$ for the depression of the transition temperature $T_\lambda - T_\lambda(Q) \sim Q^x$.

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The study of dynamic critical phenomena over the past two decades has been focused primarily on equilibrium properties in the linear-response regime [1] (except for a few studies on nonlinear relaxation [1,2]) as well as on those nonequilibrium properties where mean-field theory provides an adequate description in some respects [3]. Very little is known about nonlinear effects due to finite perturbations that bring the system out of equilibrium without implying mean-field critical behavior. Of particular interest are systems with reversible couplings since they have divergent transport coefficients [1], and thus their crossover from the linear- to the nonlinear-response regime is dominated by large fluctuation effects which may be accessible to experimental observation. In this Letter we present the first renormalization-group (RG) study on an observable divergent transport coefficient in the nonlinear-response regime where mean-field theory fails even in a qualitative sense.

A well-suited candidate for this study is the thermal conductivity $\lambda_T(Q_0)$ of bulk ^{4}He near T_λ in the presence of a finite heat current Q_0. The critical behavior of $\lambda_T(0)$ is accurately known both experimentally [4] and theoretically [5]. No theory is available so far for the nonlinear effect $\lambda_T(Q_0)-\lambda_T(0)$ apart from mean-field and scaling considerations [6]. We shall see that this effect results in a depression of the critical temperature dependence of λ_T. This is intimately related to the depression of the transition temperature $T_\lambda(Q_0)$ by a finite heat current that has been recently observed [7]. Furthermore, we predict a critical divergence $\lambda_T(Q_0) \sim Q_0^{-y_\lambda}$ for $Q_0 \to 0$ in the nonlinear-response regime near T_λ with an effective exponent $y_\lambda \approx 0.31$.

Since there exists no critical behavior of λ_T at the mean-field level a RG treatment of the fluctuations is necessary. Unlike the homogeneous situation for $Q_0 \to 0$, the problem at finite Q_0 is highly nontrivial because of the spatial inhomogeneity. Nevertheless, we are able to perform a renormalized perturbation calculation and to make a quantitative prediction on $\lambda_T(Q_0)$ without an adjustment of parameters.

We start from the Langevin equations for the order parameter $\psi(\mathbf{x},t)$ and the entropy variable $m(\mathbf{x},t)$ of model F (Ref. [1]) in the presence of a heat source W_0:

$$\dot\psi = -2\Gamma_0 \frac{\delta H}{\delta \psi^*} + i g_0 \psi \frac{\delta H}{\delta m} + \Theta_\psi , \tag{1}$$

$$\dot m = \lambda_0 \nabla^2 \frac{\delta H}{\delta m} + g_0 \nabla \cdot \mathbf{j}_s + W_0 + \Theta_m , \tag{2}$$

$$H = \int d^d x \left(\tfrac{1}{2} r_0 |\psi|^2 + \tfrac{1}{2} |\nabla \psi|^2 + \tilde u_0 |\psi|^4 + \tfrac{1}{2} \chi_0^{-1} m^2 + \gamma_0 m |\psi|^2 \right) , \tag{3}$$

with $\mathbf{j}_s(\mathbf{x},t) \equiv \mathrm{Im}[\psi^*(\mathbf{x},t) \nabla \psi(\mathbf{x},t)]$. A stationary heat current Q_0 in the z direction is produced by

$$W_0(z) = Q_0 [\delta(z+z') - \delta(z-z')] , \tag{4}$$

i.e., by a heat source and sink in the planes $z=-z'$ and $z=z'$, respectively. Eventually we shall let $z' \to \infty$. A possible Q_0 dependence of the Langevin forces is neglected. We shall always consider the stationary case. We may interpret $\delta H/\delta m = \chi_0^{-1} m + \gamma_0 |\psi|^2$ as a fluctuating local temperature variable [8] and introduce the temperature profile $T(z,Q_0)$ via the local reduced temperature

$$t(z,Q_0) = [T(z,Q_0) - T_\lambda]/T_\lambda = (r_0 - r_{0c})/2\gamma_0 \chi_0 + \langle \delta H/\delta m \rangle , \tag{5}$$

where T_λ denotes the transition temperature at $Q_0 = 0$. Taking the average of Eq. (2) and integrating over z leads to

$$\lambda_0 \frac{\partial}{\partial z} t(z,Q_0) + g_0 \langle j_s \rangle (z,Q_0) + Q_0 = 0 . \tag{6}$$

Our concept is to calculate $\langle j_s \rangle$ and to integrate Eq. (6) to obtain the stationary profile $t(z,Q_0)$. The latter can then be inverted to get $z = z(t,Q_0)$. We define a local thermal conductivity by

$$\lambda_T(z,Q_0) = -Q_0 \left(\frac{\partial}{\partial z} t(z,Q_0) \right)^{-1} \tag{7}$$

$$= \lambda_0 [1 + g_0 \langle j_s \rangle (z,Q_0)/Q_0]^{-1} \tag{8}$$

and substitute $z = z(t,Q_0)$. This yields the temperature dependence of the nonlinear thermal conductivity

$\lambda_T[t,Q_0]=\lambda_T(z(t,Q_0),Q_0)$ at finite Q_0.

The remaining task is the calculation of $\langle j_s\rangle(z,Q_0)$. In the following we confine ourselves to $T(z,Q_0)\gtrsim T_\lambda$. In the mean-field approximation corresponding to $g_0=\gamma_0=\tilde{u}_0=0$, Eqs. (1)-(4) yield

$$\langle m\rangle_{\rm MF}=-\chi_0(Q_0/\lambda_0)(z-z_0) \tag{9}$$

and $\langle\psi\rangle_{\rm MF}=0$. Accordingly, we decompose $m=\langle m\rangle_{\rm MF}+\delta m$. Treating the fluctuations $\psi(\mathbf{x},t)$ and $\delta m(\mathbf{x},t)$ up to one-loop order we obtain

$$\langle j_s\rangle(z,Q_0)=\mathrm{Im}\left[\frac{\partial}{\partial z}\langle\psi(\mathbf{x},t)\psi^*(\mathbf{x}',t)\rangle^0|_{\mathbf{x}'=\mathbf{x}}\right], \tag{10}$$

where the superscript index 0 of the dynamic propagator $\langle\psi\psi^*\rangle^0\equiv G^0$ means that $\psi,\delta m$ and the corresponding response fields $\tilde{\psi},\delta\tilde{m}$ are kept only up to second order in the dynamic statistical weight. The difficulty of the problem is due to the spatial dependence of the temperature variable

$$\tilde{r}_0(z,Q_0)=r_0+2\gamma_0\langle m\rangle_{\rm MF}=-2\gamma_0\chi_0Q_0z/\lambda_0 \tag{11}$$

already at the mean-field level according to Eqs. (5) and (9). Here we have chosen the arbitrary constant z_0 such that $\tilde{r}_0$ vanishes at $z=0$. This choice fixes the nonlinear-response region $T(z,Q_0)\approx T_\lambda$ of the temperature profile at the origin of the z axis for arbitrary Q_0, which is well adapted to the investigation of the critical Q_0 dependence. Expanding G^0 with respect to Q_0 at fixed $\tilde{r}_0$ leads to a series that, for our case $\mathbf{x}'=\mathbf{x}$, can be summed up exactly. The details of this calculation will be published elsewhere. The result is, in three dimensions,

$$\langle j_s\rangle(z,Q_0)=-\tfrac{1}{2}(Q_0g_0/\lambda_0\Gamma_0')\Phi_{1/2}(X_0)\int_k[\tilde{r}_0+k^2]^{-2}, \tag{12}$$

where

$$\Phi_p(y)=\Gamma(p)^{-1}\mathrm{Re}\left[(-y)^{-p/3}\int_0^\infty ds\,s^{p-1}\exp[-s^3-s(-y)^{-1/3}]\right] \tag{13}$$

with $(-y)^{-1/3}=y^{-1/3}e^{i\pi/3}$ for $y>0$ and $\Phi_p(0)=1$. At $X_0=0$, Eq. (12) yields the one-loop term of the linear thermal conductivity [5]. The nonlinear effect is contained in

$$X_0=-\frac{1}{6}\left[\frac{Q_0}{\lambda_0}\right]^2\tilde{r}_0^{-3}\left[\frac{\frac{1}{8}g_0^2-\gamma_0\chi_0g_0\Gamma_0''}{\Gamma_0'^2}-2\gamma_0^2\chi_0^2\right]. \tag{14}$$

An appropriate description of the critical behavior requires renormalization of the bare perturbative result (12)-(14). The renormalizations at $Q_0=0$ are well known [5,9]. Since the ultraviolet divergences are not changed by a finite heat current no new renormalizations are necessary; thus we can express our results in terms of the known effective parameters [5,9] $v[t]$, $v=(w,\gamma,F,f,u)$, at $Q_0=0$. The argument t, however, is replaced by a Q_0-dependent flow parameter $\tau(z,Q_0)$ which is determined implicitly by

$$r_\tau(z,Q_0)+[Q_0\xi(\tau)^2/g_0]^2=1. \tag{15}$$

Here

$$r_\tau(z,Q_0)=-8\pi\gamma[\tau]F[\tau][Q_0\xi(\tau)^2/g_0]z/\xi(\tau) \tag{16}$$

is the renormalized counterpart of the temperature variable $\tilde{r}_0$ [Eq. (11)], with $\xi(t)$ being the correlation length above T_λ (Ref. [4]). From (6) and (12)-(14) we obtain the reduced temperature profile,

$$t(z,Q_0)=\tau r_\tau\{1-\tfrac{1}{2}f[\tau][r_\tau^{-1/2}\Phi_{-1/2}(X_\tau)-1]\}, \tag{17}$$

$$X_\tau=-\tfrac{1}{3}\pi^2f[\tau]r_\tau^{-3}[Q_0\xi(\tau)^2/g_0]^2(f[\tau]-8\gamma[\tau]F[\tau]w''[\tau]/w'[\tau]-16\gamma[\tau]^2w'[\tau]), \tag{18}$$

where X_τ is the renormalized counterpart of X_0. Equation (15) can be combined with (17) to eliminate r_τ and to get $\tau=\tau[t,Q_0]$. From (8) and (12)-(14) we then obtain the nonlinear thermal conductivity,

$$\lambda_T[t,Q_0]=\frac{g_0[\xi(\tau)k_BC_P(\tau)]^{1/2}}{2\pi^{1/2}F[\tau]\{1+\gamma[\tau]^2F_+(u[\tau])\}^{1/2}}\{1-\tfrac{1}{2}f[\tau][\tfrac{1}{2}r_\tau^{-1/2}\Phi_{1/2}(X_\tau)-1]\}^{-1}, \tag{19}$$

with $C_P(t)$ being the specific heat.

Equations (15)-(19) constitute the main results of this paper. They reveal the following three general features.

(i) A finite heat current drives the system away from criticality, i.e., $\tau>0$, since (15) and (17) imply a finite, Q_0-dependent correlation length $\xi[t,Q_0]\equiv\xi(\tau[t,Q_0])$.

(ii) For $T\approx T_\lambda$ (or $r_\tau\approx 0$) the inverse length scale $\xi[0,Q_0]^{-1}\sim(Q_0/g_0)^{1/2}$ constitutes the basic measure for the distance from criticality ($g_0=2.2\times10^{11}$ sec^{-1}, Q_0 has the units cm^{-2}sec^{-1}).

(iii) The linear and nonlinear critical regimes are identified as $|X_\tau|\ll 1$ and $|X_\tau|\gtrsim 1$, respectively; in the t-Q_0 plane this means $t\gg t_c$ and $t\lesssim t_c$ with $t_c(Q_0)$ given in Eq. (21) below. This is illustrated in Fig. 1 together with the range where previous experiments [4,7] have been performed.

The measurable heat current Q (in units of W/cm^2) is related to our Q_0 by $Q=k_BT_\lambda Q_0$. In Fig. 2 we have plotted [10] $\lambda_T[t,Q_0]$. We see that a finite Q implies a depression of the critical temperature dependence. In the range $Q\gg 0.07$ μW/cm^2 this depression dominates the

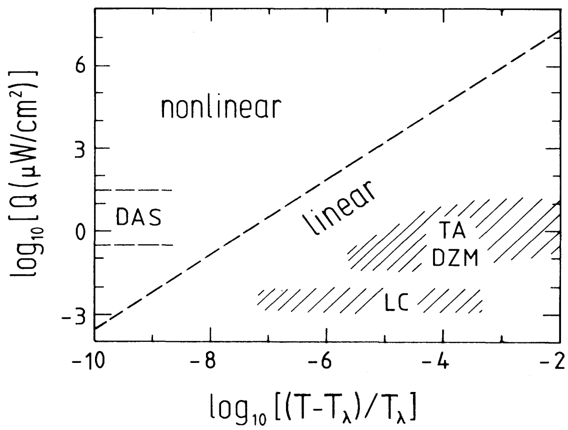

FIG. 1. Linear and nonlinear critical regimes in the t-Q plane. The dashed line corresponds to $X_\tau \approx 1$, $t \approx t_c(Q_0)$. The range of previous experiments is indicated by the shaded areas (TA, DZM, LC, Ref. [4]) and by the horizontal dashed lines (DAS, Ref. [7]).

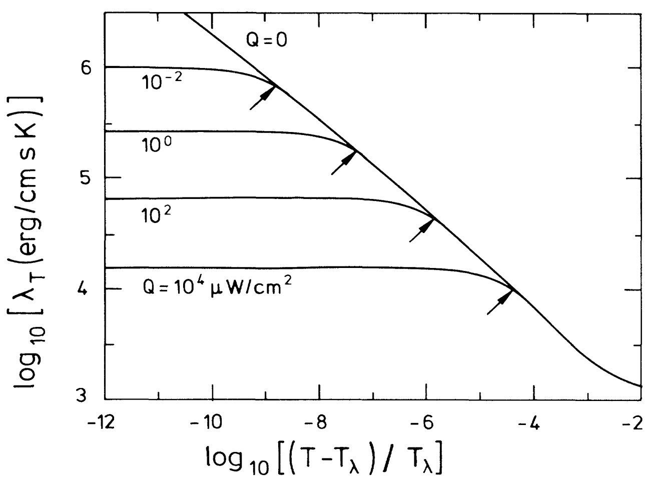

FIG. 2. Theoretical prediction [Eq. (19)] for the thermal conductivity vs reduced temperature for various values of the heat current Q. The arrows indicate the crossover temperature t_c [Eq. (21)].

conventional gravity-induced rounding which will be further discussed below. The temperature profile, Eq. (17), can be written in the quasiscaling form

$$t(z,Q_0)=qG_Q(s), \quad s=8\pi\gamma[q]F[q]Q_0^{1/2}g_0^{-1/2}z\,, \tag{20}$$

with $q\equiv(Q_0\xi_0^2/g_0)^{1/2\nu}$ and $\xi_0=1.4$ Å. If nonasymptotic effects of the effective parameters $v[t]$ were negligible $G_Q(s)$ would be universal, i.e., independent of Q. The weak-scaling fixed point [5] and the slow approach of the specific heat to its finite value at criticality, however, imply a weak nonuniversal Q dependence of $G_Q(s)$ in Fig. 3. The finite heat current causes a finite temperature gradient at $T=T_\lambda$ [finite slope $G_Q'(0)$]. This corresponds to a finite value of $\lambda T[0,Q_0]$ in Fig. 2.

One may define a crossover temperature $t_c(Q_0)$ at which $\lambda_T[t,Q_0]$ starts to deviate significantly (by more than, say, 5%) from $\lambda_T[t,0]$ (arrows in Fig. 2). In the range $10^{-10}\lesssim Q\lesssim 10^2$ W/cm^2 our results can be represented as

$$t_c(Q_0)=A_t(Q_0\xi_0^2/g_0)^{x_t}\,, \tag{21}$$

$$\lambda_T[t_c,Q_0]=A_\lambda g_0\xi_0^{-1}k_B(Q_0\xi_0^2/g_0)^{-y_\lambda}\,, \tag{22}$$

with the effective exponents $x_t\approx 0.74\approx(2\nu)^{-1}$ and $y_\lambda\approx x_\lambda/2\nu\approx 0.31$, where $x_\lambda\approx 0.42$ is the effective exponent of $\lambda_T[t,0]\sim t^{-x_\lambda}$ for $10^{-8}\lesssim t\lesssim 10^{-4}$. The same exponent y_λ describes the divergence of $\lambda_T[0,Q_0]\sim Q_0^{-y_\lambda}$. Our one-loop results for the amplitudes are $A_t\approx 2.8$ and $A_\lambda\approx 0.046$ (Ref. [10]).

An experimental verification of these predictions would constitute an important test of the RG theory in the nonlinear-response regime. In deriving these predictions we have neglected the effect of gravity which causes a spatial variation

$$|\partial T_\lambda/\partial z|=\rho g|\partial T_\lambda/\partial P|=1.3\times 10^{-6}\ \text{K/cm} \tag{23}$$

of T_λ at $Q=0$ (Ref. [11]). Thus this effect will mask the nonlinear temperature gradient $\partial T(z,Q_0)/\partial z$ induced by the heat current if $|\partial T(z,Q_0)/\partial z|\lesssim|\partial T_\lambda/\partial z|$ in the region $T(z,Q_0)\approx T_\lambda$. From our results we estimate that in an Earth-bound experiment this gravity effect on the nonlinear part of $T(z,Q_0)$ is negligible for $Q\gg 0.07\ \mu$W/cm^2 where $|\partial T/\partial z|\gg|\partial T_\lambda/\partial z|$.

This regime has already been realized experimentally [7] (compare Fig. 1) but the detailed form of the temperature profile $T(z,Q_0)$ was not detected. According to Fig. 3, the nonlinear part of $T(z,Q_0)$ varies on the scale of $O(10^{-3}$ cm) for a typical heat current $Q=1\ \mu$W/cm^2,

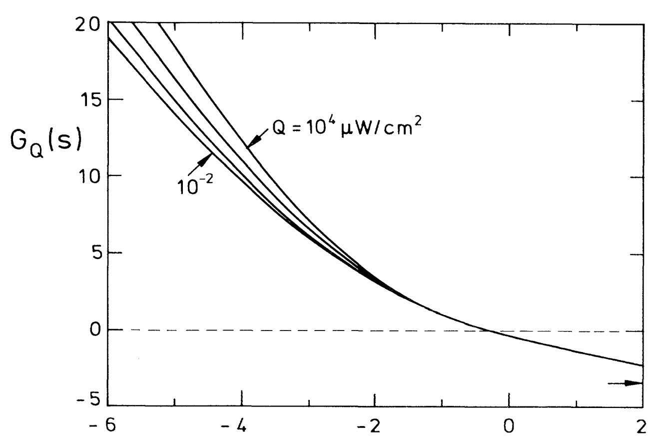

FIG. 3. Quasiscaling function $G_Q(s)$ of the reduced temperature profile $t(z,Q_0)$, Eqs. (5), (17), and (20), for various Q corresponding to Fig. 2. The nonlinear region is $s\gtrsim -1.5$. The dashed line indicates $T=T_\lambda$. The arrow indicates $G_Q(\infty)$ corresponding to $t(\infty,Q_0)$.

and thus the requirement on the spatial resolution is rather demanding. Here we propose an approach that primarily needs high-resolution thermometry rather than high spatial resolution. Consider the temperatures $T_1 \equiv T(z_1,Q_0)$ and $T_2 \equiv T(z_2,Q_0)$ measured at two fixed positions z_1 and z_2. Suppose that T_1 and T_2 are changed by δT_1 and δT_2 such that Q remains unchanged and that δT_2 and δT_1 are measured. This would determine the derivative $(\partial T_2/\partial T_1)_Q \approx \delta T_2/\delta T_1$ at fixed Q. According to Eq. (7), this yields

$$\left(\frac{\partial T_2}{\partial T_1}\right)_Q^{-1} = \frac{\lambda_T[t_2,Q_0]}{\lambda_T[t_1,Q_0]}, \tag{24}$$

with $t_i = t(z_i,Q_0)$, $i=1,2$. The basic idea is to choose $|z_2 - z_1| \gg (g_0/Q_0)^{1/2}$ such that T_1 is in the linear critical regime well above T_λ whereas T_2 is in the nonlinear vicinity of T_λ. Since $\lambda_T[t_1,Q_0] \approx \lambda_T[t_1,0]$ is well known [4,5] the measurement of $\partial T_2/\partial T_1$ determines the nonlinear thermal conductivity $\lambda_T[t_2,Q_0]$ according to Eq. (24). We estimate that for δT_2 a temperature resolution (in units of K) of about $10^{-8} \times Q^{3/4}$ (Q in units of μW/cm^2) is needed. Perturbing effects due to the boundary resistance may be avoided in a cell with double midplane thermometers being planned by Ahlers [11]. Experiments in space would be advantageous in that they could explore $t(z,Q_0)$ at very small Q_0 where the nonlinear portion of $t(z,Q_0)$ varies on a macroscopic length scale $(g_0/Q_0)^{1/2}$, for example, $(g_0/Q_0)^{1/2} = 0.1$ mm for $Q = 0.06$ μW/cm^2.

Our result for $t(z,Q_0)$, Eq. (17), is valid for $z \le 0$ but remains applicable also to $0 < z \lesssim O(g_0^{1/2} Q_0^{-1/2})$ slightly below T_λ (Fig. 3) where the spatial variation of the order parameter $\langle \psi \rangle (z)$ is still negligible. It would be interesting to extend our calculation to the entire interface region $z \gtrsim (g_0/Q_0)^{1/2}$. Here we report on an exact result in the limit $z \to \infty$. Apart from the effect of vortices we have determined the exact exponent of

$$t(\infty,Q_0) = -A_\infty (Q_0 \xi_0^2/g_0)^x$$

as

$$x = [(d-1)\nu]^{-1}, \tag{25}$$

and thus $x \approx 0.74$ in $d=3$ dimensions. For the amplitude we have found $A_\infty = 3.2$ in one-loop order (arrow in Fig. 3). Equation (25) can be derived from dimensional analysis and from the fact that Q_0 and g_0 are renormalized by the same Z factor $Z_m^{-1/2}$ (Ref. [5]). Equation (25) is consistent with one of the results by Onuki [6,12]. Although vortex generation may become important at finite Q it seems to be justified to compare our result (25) with the measured [7] exponent $x^{\text{expt}} = 0.813 \pm 0.012$, since in this experiment the thermal gradient due to vortices in the superfluid appeared to be negligible [7]. Nevertheless, the discrepancy between (25) and x^{expt} may partially be due to the effect of vortices. Further experimental and theoretical work on the profile $t(z,Q_0)$ would be desirable.

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