# Frustration in a Linear Array of Vortices

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The experimental study of a linear array of vortices shows the existence of spatially modulated states of oscillation above onset. A model of coupled oscillators is proposed, and the measurements of the corresponding first- and second-neighbor couplings show that these regimes are the result of frustration.

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These last years, several experimental studies of spatiotemporal chaos have been performed on extended systems whose dynamics evolves in a one-dimensional space. Such systems display a very rich dynamical behavior, and, because of their low dimensionality, they remain accessible to theoretical modeling. It has thus been possible to interpret chaotic regimes, such as spatiotemporal intermittency [1], dispersive chaos [2], and various phenomena, such as parity breaking [3], within the framework of one-dimensional amplitude equations. Because of their reasonable complexity, one can hope that one-dimensional systems are good candidates for investigating new routes to weak turbulence. In this Letter, we consider a particular system of this type—a line of vortices—and we report, for the first time, the observation of frustrated states. Frustrated states have been studied for many years in other contexts, such as spin glasses, but, to the best of our knowledge, their existence has never been reported in hydrodynamic systems.

The experimental system that we use is similar to that described in a recent Letter [4]. The cell, made of polyvinylchloride (PVC), is 450 mm long, 40 mm wide, and 50 mm high. In the bottom of the cell, a groove 300 mm long, 20 mm wide, and 3 mm deep is machined. Outside the cell, and just below the groove, a line of alternating magnets, $5\times8\times3$ mm in dimension, is formed. Each individual magnet produces a maximum magnetic field of 0.3 T. The cell is filled with a normal solution of sulfuric acid, at a level corresponding to the depth of the groove so as to suppress the meniscus along the boundary of the lattice. We impose a steady electric current along the cell, and the resulting magnetic forces, which are spatially periodic, induce recirculating flows; we thus obtain a linear array of counterrotating vortices, 5 mm wide and 20 mm long, whose number can be varied from a pair to 36.

Various conditions at the extremities of the lattice have been considered. Most of the results presented herein have been obtained by incorporating additional smaller magnets at the ends of the magnetic line; by doing this, we induce weak recirculating flows at the extremities of the lattice, which in turn force the end vortices' centers to fall, on average, close to the lattice axis. Reducing lattice distortions at the extremities turned out to be crucial for performing dynamical measurements on small systems.

As shown in previous studies [4,5], as the electric current is increased, the system undergoes a transition from a state composed of counterrotating vortices to a state where all the vortices have the same sign and are 2 times larger (the latter state is called "state +"). This state will be further subjected to temporal instabilities. We use the shadowgraph method to visualize the separatrices between the vortices, and thus follow the spatiotemporal dynamics of the system (see Fig. 1, which represents the shadowgraph image of a system of fifteen corotating vortices). For the measurements, the shadowgraph images are digitized and the positions of the separatrices are determined by tracking, in real time, the maxima of the light intensity along the lattice axis. The resulting signal-to-noise ratio is about 40 dB.

In the case of four magnets, state + is a single pair of steady corotating vortices, and the bifurcation to the monoperiodic regime appears at $I\approx15$ mA. As in previous studies [4,5], we observe that the bifurcation is supercritical, and the corresponding Hopf oscillator is stable within a large range of variation of $I$ above threshold. For the case of six magnets, state + is composed of three corotating vortices, and the system sustains two Hopf oscillators. The bifurcation to the oscillatory regime appears at $I\approx13$ mA and above onset, the two oscillators have the same amplitude and are in phase, so that the collective oscillation of this small system has the form of

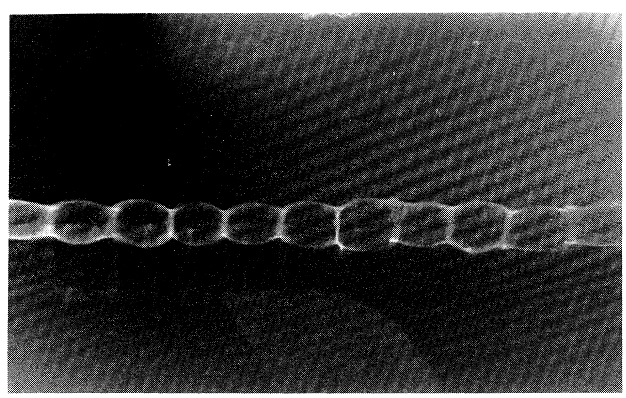

FIG. 1. Shadowgraph image of the free surface of the flow for a system of fifteen corotating vortices, for $I=15$ mA (only a part of the lattice is shown).

 

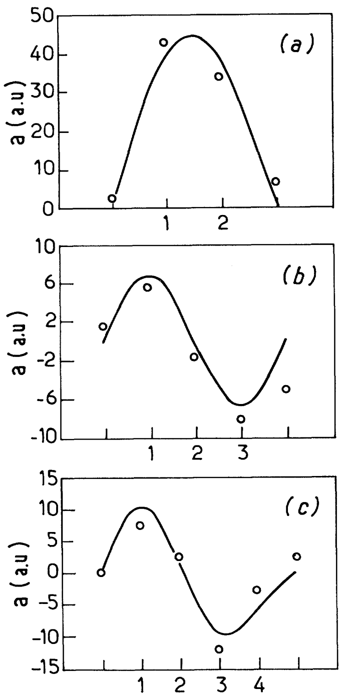


FIG. 2. Spatial structures of the amplitudes $a$ of oscillation of the separatrices, close to the onset, for systems of small sizes. $n$ is the position of the separatrix along the lattice. The amplitude is taken positive when the temporal phase is close to zero, and negative when it is close to 180° (here, the reference of phase is the first oscillator). The corresponding values of the current are (a) $I=15.73$ mA (for $N=2$), (b) $I=13.95$ mA (for $N=3$), and (c) $I=13$ mA (for $N=4$). The full lines are calculated using Eq. (1).

an acoustical mode [see Fig. 2(a)]. A remarkable structure of the oscillatory regime appears for the case of eight magnets, i.e., three oscillators. In this case, close to the onset of oscillation, the two extreme oscillators have the same amplitude and are out of phase by 180°, whereas the central oscillator is damped [see Fig. 2(b)]. This defines a mode of oscillation of wave number $k=\pi/2$, which turns out to be stable over a large range of values of $I$ above onset. Thus in this elementary system, one observes, as the first mode of instability, a spatially modulated state. The case $N=4$ shows a similar feature [see Fig. 2(c)].

In general, deep modulations of amplitude are observed in lattices of larger sizes. This feature is clearly visible on the typical direct time recording of a large lattice (see Fig. 3). Figure 4 shows corresponding plots of the amplitudes and phase along the lattice for systems of eight and fifteen oscillators. We have "domains," which are separated by deep minima of amplitude [see Figs. 4(a) and 4(c)], moreover, one or two waves, with wave numbers lying in the range 80°–120°, propagate along the lattice [see Figs. 4(b) and 4(d)]. Depending on the initial conditions, one can have either two counterpropagating waves propagating from the ends towards a sink [see Fig. 4(b)] or a single wave along the lattice [see Fig. 4(d)]. The wave number of such waves increases with $I$, from $80° \pm 10°$ per unit of oscillator at threshold up to $135° \pm 20°$ at the onset of the chaotic regime, indicating the probable existence of a selection process. As shown in Figs. 4(a) and 4(c), the sizes of the domains are not regular—they range from two to four oscillators—and their positions also depend, in an uncontrolled way, on the history of the system. As a general tendency, the domain sizes, on average, decrease as the current increases. The spatially modulated regime becomes chaotic as $I$ is increased above 20 mA. In our system, spatiotemporal chaos can be crudely characterized by the fact that the boundaries of the domains evolve erratically in space.

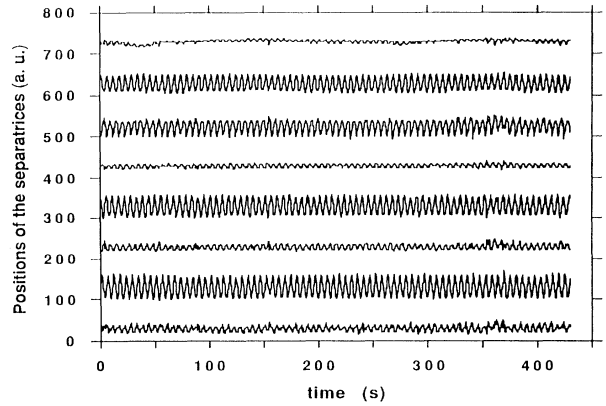


FIG. 3. Direct time recordings of the positions of eight separatrices in the central part of a system of fifteen oscillators, for $I=14$ mA.

Owing to the geometry of our experiment, and to the fact that one can define an oscillator for an isolated pair

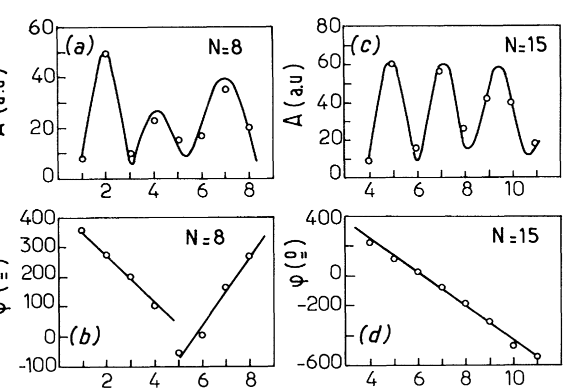


FIG. 4. Typical evolutions of the amplitudes and phases with the position of the separatrices, for systems of large sizes. (a),(b) $N=8$, $I=16.04$ mA; (c),(d) $N=15$, $I=18.03$ mA (in the latter case, only a part of the system is shown). The full lines are drawn to guide the eye.

of vortices, it is tempting to view our system as a chain of coupled oscillators. As in a previous study [4], each element of the chain is a supercritical Hopf oscillator, coupled linearly and nonlinearly to its first neighbor. However, we introduce now a second-neighbor coupling, which we assume, for simplicity, to be linear. We then get the following system of equations:

$$\frac{\partial W_n}{\partial t} = \mu(1+ic_0)W_n - (1+ic_2)W_n|W_n|^2 + \varepsilon(1+ic_1)(W_{n-1}+W_{n+1}) - (c_3+ic_4)W_n(|W_{n-1}|^2+|W_{n+1}|^2) + \varepsilon'(1+ic_1')(W_{n-2}+W_{n+2}). \quad (1)$$

Here, $W_n$ is the complex amplitude of the $n$th oscillator, $\mu$ is the control parameter, and $c_0$, $c_1$, $c_2$, $c_3$, $c_4$, $c_1'$, $\varepsilon$, and $\varepsilon'$ are real numbers.

The linear modes of instability of the chain are traveling waves in the form $W_n = W_0 e^{i(\omega t + kn)}$. When $\varepsilon'$ is positive, the critical modes are either acoustical (for $\varepsilon$ positive) or optical (for $\varepsilon$ negative), as in the case $\varepsilon'=0$. An interesting situation arises when the second-neighbor-coupling coefficient $\varepsilon'$ is negative: In this case, first-neighbor coupling favors each oscillator being in phase with its second neighbor, whereas second-neighbor coupling favors a phase shift of 180° between them. In our problem, as in magnetic systems, this competition generates frustration if $|\varepsilon'/\varepsilon|$ is larger than a critical value, which is found to equal $\frac{1}{4}$ for infinite lattices. When this condition is satisfied, the critical mode is neither acoustical nor optical, but a traveling wave with a well-defined wave number $k_0$ given by the relation

$$\cos k_0 = -\varepsilon/4\varepsilon'. \quad (2)$$

Indeed, for finite lattices with reflecting boundary conditions, linear theory predicts a standing wave at onset; in this case, the scale $\lambda_0 = \pi/k_0$ represents a typical distance between two successive nodes. It is interesting to note that condition (2) is formally identical to that characterizing the onset of helimagnetic structures in models including only first- and second-neighbor coupling [6].

In order to make a quantitative comparison between this model and our experiment, we proceed to the measurement of all the coefficients involved in Eq. (1). Let $N$ be the number of oscillators, the separatrices are labeled from 1 to $N$, and $W_n$ is related to the temporal behavior of the $n$th separatrix. The coefficients $c_0$, $c_1$, $c_2$, $c_3$, $c_4$, $\varepsilon$, and $\mu=(I-I_c)/I_c$ (where $I_c$ is the critical value of the electric current for $N=1$) are determined by investigating threshold values and critical properties in systems composed of one and two oscillators. We thus obtain the following values: $I_c \approx 15$ mA, $c_0 \approx -0.11$, $c_1 \approx 0.22$, $c_2 \approx -1.34$, $c_3 \approx 3$, $c_4 \approx 0.046$, and $\varepsilon \approx 0.23$. For the second-neighbor-coupling coefficients $c_1'$ and $\varepsilon'$ we consider the case $N=3$, for which, as shown in the experiment, the excited wave number is $\pi/2$. Using this value, we determine the theoretical values of the electric current and the frequency at onset (obtained under the condition that there is no oscillation outside the lattice). We further compare them to the experimental values, and thus obtain $c_1' \approx 0.22$ and $\varepsilon' \approx -0.23$. According to this, $\varepsilon'$ is negative, the ratio $|\varepsilon'/\varepsilon|$ is larger than 1, and therefore we are effectively in a frustrated situation. The wave number determined by relation (2) is found to equal 76° (per oscillator), so that $\lambda_0 \approx 2.4$.

Once all the coefficients of Eq. (1) are determined, we perform a numerical simulation of system (1) by using a fourth-order Runge-Kutta method. For the boundary conditions, we assume no oscillation outside the lattice. The results obtained for various system sizes are shown in Figs. 2 and 5. As expected, close to the onset, the states of oscillation of the experiment is well reproduced by the numerical simulation: We successively obtain the acoustical mode for $N=2$ and the antisymmetric ones for $N=3$ and 4 (see the full lines in Fig. 2). For large systems, we find that the oscillation amplitude is modulated in space, with domains of typically two or more oscillators; this is the typical form of the frustrated states for our system. Actually, in the simulation, as $\mu$ is varied, one gets two distinct modulated structures: Close to the onset $\mu_c \approx -0.45$, the states of oscillation have a spatial symmetry (symmetric or antisymmetric according to the parity of the number of oscillators), then, for values of $\mu$ above $\mu_1 \approx -0.23$, the states of oscillation are asymmetric. Figure 5 shows the results of the numerical study for $N=8$ and 15, for values of $\mu$ close to those of Fig. 4, i.e., $\mu=-0.05$ and $-0.2$ (for smaller values of $\mu$, the signal was within the experimental noise). In both cases we obtain asymmetric states with deep modulations of amplitude, and domains of two or more oscillators [see

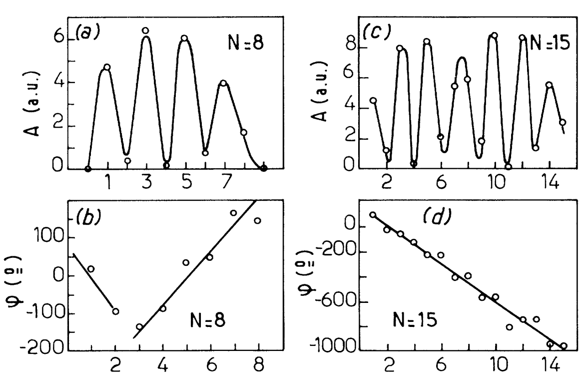


FIG. 5. Numerical results showing typical evolutions of the amplitudes and phases along a chain of oscillators, for systems of large sizes. (a),(b) $N=8$, $\mu=-0.05$; (c),(d) $N=15$, $\mu=-0.2$ (in the latter case, only a part of the system is shown). The full lines are drawn to guide the eye.

Figs. 5(a) and 5(c)]. Concerning the phase, typical results of the simulation are shown in Figs. 5(b) and 5(d). For such plots, as in the experiment, we smooth out the curve by adding multiples of 360° (this procedure may be somewhat artificial when the jump between two oscillators is close to 180°). Depending on the initial conditions, we obtain either a counterpropagative wave [see Fig. 5(b)] or a single traveling wave [see Fig. 5(d)]. These features are in good agreement with the observations. Moreover, such waves are characterized by a wave number $k_0$ equal to 76° at threshold, which increase up to 120° just before the chaotic state; such values are consistent with those obtained experimentally. Finally, in large systems, one gets spatiotemporal chaos as $\mu$ is increased. The qualitative features of this weakly turbulent regime also show similarities with the experiment; in short, it consists of the erratic wandering of nodes along the lattice.

In conclusion, we get good agreement between the frustrated model (1) and experiment for small and large systems, showing that the structure of the oscillatory states observed in the experiment is the result of a frustration. The observation of frustrated states has not been reported previously in hydrodynamics [7], but one can mention that succession of nodes and antinodes has also been observed, close to the onset, in a linear array of trapped convective rolls [8]. As for our system, they may be the remnants of a frustration and this suggests that this type of effect may have some general relevance in discrete systems.

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[7] In the experiment quoted in Ref. [4] (for which the fluid layer thickness was 2 mm), there was no clear evidence for the existence of frustration; we further found that the second-nearest-neighbor coupling is effectively weaker in this case, and that, as a rough approximation, frustration could be ignored, at least for the lattice sizes which were considered.

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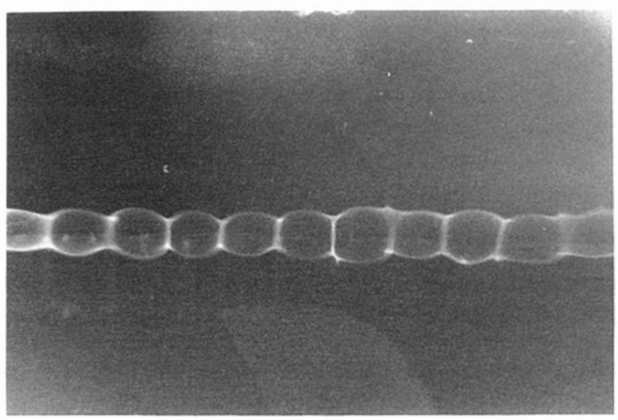

FIG. 1. Shadowgraph image of the free surface of the flow for a system of fifteen corotating vortices, for $I = 15$ mA (only a part of the lattice is shown).