Nucleon-Nucleon Potential in the Skyrme Model: Beyond the Product Approximation

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Exact numerical calculations with Lagrange constraints are used to determine the lowest terms in an expansion for the two-Skyrmion interaction. The nucleon-nucleon potential which results after semiclassical quantization compares better with the phenomenological Paris potential than do previous calculations in the Skyrme model which used the product approximation. In particular, the present calculations show a sizable medium-range attraction in the central channel, a result that is not found with the product approximation.

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There have been several calculations over the past few years of the nucleon-nucleon potential in the Skyrme model [1-5], all of which use the product approximation to simplify the numerics. With this approximation the two-Skyrmion potential as a function of the relative angles of orientation between the Skyrmions has a compact form, and the extraction of the nucleon-nucleon potential by projection onto asymptotic two-nucleon states is straightforward. This gives three nonvanishing channels, the central, spin-spin, and tensor; the latter two compare well at large and intermediate distances with the phenomenological Paris potential [6]. The major inadequacy previous calculations have revealed is the lack of an intermediate-range attraction in the central potential. Although many remedies have been proposed, this result may not be due to a fault of the Skyrme model. As mentioned in the original paper by Jackson, Jackson, and Pasquier [1], the product approximation, which is not a solution to the equations of motion, can only be considered accurate at large distances; and the failure of these calculations to reproduce the central-range attraction may simply be the failure of the product approximation to provide an adequate approximation to the exact solution. Indeed, the symmetrized product approximation, which is designed to respect the symmetries of the exact solutions (the simple product approximation respects these symmetries only in the asymptotic limit), was used in a calculation by Nyman and Riska [4], and a small attraction was found. (As shown in Ref. [7], however, this approximation is inadequate at short range since the baryon number becomes ambiguous.) In this Letter, we go further by studying the two-Skyrmion potential through exact numerical calculations; this allows a more meaningful comparison between the Skyrme model and phenomenological nucleon-nucleon potentials.

We use here the Skyrme model with the original stabilizing term fourth order in derivatives and a pion mass term. Expressing the pion field via the SU(2) matrix $U(\mathbf{x},t)=\exp[(i/f_\pi)\boldsymbol{\tau}\cdot\boldsymbol{\pi}(\mathbf{x},t)]$, the Lagrange density has the following form:

$$\mathcal{L}=\frac{f_\pi^2}{4}\mathrm{Tr}(\partial_\mu U\partial^\mu U^\dagger)+\frac{1}{32e^2}\mathrm{Tr}[\partial_\mu U U^\dagger,\partial_\nu U U^\dagger]^2+\frac{f_\pi^2 m_\pi^2}{2}\mathrm{Tr}(U-1)\,. \quad (1)$$

The constants f_π and e are here taken to be 93 MeV and 4.76 so as to fit the zero-baryon sector [1]. By virtue of the Goldberger-Treiman relation, this parameter set guarantees that the long-range part of the potential —which automatically in the Skyrme model is due to one-pion exchange— will agree with phenomenological results. This requires the introduction of finite-N_c corrections (N_c denotes the number of quark colors) described in Ref. [1].

The single-baryon solutions of minimum energy have the hedgehog form

$$U_H=A\exp[i\boldsymbol{\tau}\cdot\hat{\mathbf{r}}F(r')]A^\dagger\,,$$

with A a constant SU(2) matrix, $\mathbf{r}'=\mathbf{r}-\mathbf{R}$, and $\mathbf{R}$ a constant spatial vector. Naive semiclassical quantization consists of promoting the collective coordinates $\mathbf{R}$, the Skyrmion center, and A, the orientation in isospace, to quantum variables. In the c.m. frame, the resulting quantum states are given by the Wigner $\mathcal{D}$ functions:

$$\psi(N)=\sqrt{2t+1}\mathcal{D}^{(t)}_{i,-s}(A)\,,$$

with $t=j$ a half integer and i and s the third components of isospin and spin, respectively. The states of lowest energy, $t=j=\frac{1}{2}$, are the nucleon wave functions; states with $t=j=\frac{3}{2}$ correspond to the delta.

A good description of the low-energy two-baryon ($B=2$) sector in the Skyrme model can be attained by expressing the general field configurations in terms of twelve collective variables [8], which can be taken as a global translation (the center-of-mass position), the global orientations in space and isospace, the spatial separation, and the relative orientation in isospace. In the asymptotic limit, where the configurations consist of well-separated single baryons, these variables may be reexpressed as the positions $\mathbf{r}_i$ and orientations A_i of the two Skyrmions. Since these variables are independent in this limit, quantization yields asymptotic states which are simply the product of the free nucleon states derived from the hedgehog solutions. For general $B=2$ configurations, the energy is independent of the global collective coordinates; these variables describe zero modes. Thus the effective Hamiltonian which results from the canonical quantization of these collective variables will involve a potential which depends on only the separation r, taken here to be along the 3 axis, and the relative isospatial angles of orientation, which can be expressed as Euler angles via the SU(2) matrix $C=A_1^\dagger A_2=e^{i\tau_3\alpha/2}e^{i\tau_2\beta/2}e^{i\tau_3\gamma/2}$. Since the $\mathcal{D}$ functions form a complete set over SU(2) for integer j, a general form for the two-Skyrmion potential is given by

$$V(r,C)=\sum_{j=0}^{\infty}\sum_{m=-j}^{j}\sum_{n=-j}^{j}V_{jmn}(r)\mathcal{D}^{(j)}_{m,n}(C).$$

However, configurations with orientations C differing only by a rotation about the 3 axis are degenerate in energy—they are simply related by a redefinition of the 1 and 2 axes in space and isospace [8,9]. Thus the potential is independent of $\alpha-\gamma$. Furthermore, we expect $V(r,C^\dagger)=V(r,C)$; that is, that the classical potential is symmetric under Skyrmion interchange $A_1\leftrightarrow A_2$. We thus arrive at the following form for the two-Skyrmion potential:

$$V(r,C)=\sum_{j=0}^{\infty}\sum_{m=0}^{j}V_{jm}(r)\tfrac{1}{2}[\mathcal{D}^{(j)}_{m,m}(C)+\mathcal{D}^{(j)}_{-m,-m}(C)]. \tag{2}$$

The nucleon-nucleon potential is then obtained by sandwiching Eq. (2) between the asymptotic two-nucleon states given by

$$\psi(N_1,N_2)=2\mathcal{D}^{(1/2)}_{i_1,-s_1}(A_1)\mathcal{D}^{(1/2)}_{i_2,-s_2}(A_2). \tag{3}$$

Since $\mathcal{D}^{(j)}_{m,m}(C)=\sum_n\mathcal{D}^{(j)}_{m,n}(A_1^\dagger)\mathcal{D}^{(j)}_{n,m}(A_2)$, one finds that terms with $j\geq 2$ in Eq. (2) can contribute to the nucleon-nucleon potential only through intermediate states.

The product approximation, which may be written

$$U_p=A_1U_H(\mathbf{x}+\tfrac{1}{2}r\hat{\mathbf{e}}_2)A_1^\dagger A_2U_H(\mathbf{x}-\tfrac{1}{2}r\hat{\mathbf{e}}_3)A_2^\dagger,$$

generates a potential consisting only of terms with $j\leq 2$ [10]. Previous calculations have shown, however, that the coefficients V_{2m} are small with respect to V_{1m} and V_{00}, and that the potential derived in the product approximation is fitted well by truncating Eq. (2) at $j=1$ [1]. This means that one need then perform calculations for only three different orientations C in order to determine the two-Skyrmion potential. For exact calculations this would be especially helpful, and so for the present we assume that the potential is well parametrized by

$$V(r,C)=V_{00}(r)\mathcal{D}^{(0)}_{0,0}(C)+V_{10}\mathcal{D}^{(1)}_{0,0}(C)+V_{11}(r)[\mathcal{D}^{(1)}_{1,1}(C)+\mathcal{D}^{(1)}_{-1,-1}(C)]. \tag{4}$$

The main purpose of our investigation is to determine the reliability of the product approximation in approximating the two-Skyrmion solutions, and assuming the form Eq. (4) is sufficient for this goal. Sandwiching Eq. (4) between the states in Eq. (3) then gives

$$V(N_1,N_2)=V_C(r)+V_{SS}(r)\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\boldsymbol{\sigma}^{(1)}\cdot\boldsymbol{\sigma}^{(2)}+V_T(r)\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)}\sigma_i^{(1)}\cdot\sigma_j^{(2)}(3\delta_{i3}\delta_{3j}-\delta_{ij}), \tag{5}$$

with the central potential $V_C=V_{00}$, the spin-spin potential $V_{SS}=25(V_{10}+V_{11})/243$, and the tensor potential $V_T=25(2V_{10}-V_{11})/486$. As in Ref. [1], we have multiplied V_{SS} and V_T by the factor $(N_c+2)^2/N_c^2=25/9$, which is thought to arise as a finite-N_c correction.

The calculations are performed by discretizing the equations of motion on a $20\times20\times40$ spatial lattice and relaxing from an initial $B=2$ configuration. Determining the separation r is nontrivial since the soliton centers are not well-defined quantities [11]. Here we take the separation to be twice the rms radius of the baryon number density $\mathcal{B}^0$,

$$\frac{1}{4}r^2=\frac{1}{B}\int d^3x\,\mathbf{x}^2\mathcal{B}^0(\mathbf{x}).$$

For point particles,

$$\mathcal{B}^0(\mathbf{x})=\delta^3(\mathbf{x}+\tfrac{1}{2}r\hat{\mathbf{e}}_3)+\delta^3(\mathbf{x}-\tfrac{1}{2}r\hat{\mathbf{e}}_3),$$

and this reduces to the usual definition of the separation $r=|\mathbf{r}_1-\mathbf{r}_2|$. The calculations are performed for different values of r which are fixed by adding to Eq. (1) a Lagrange constraint [12]. For a given orientation C, the fields in the $z>0$ and $z<0$ halves of space are related by the rotation defined by C. For each C there is a vector $\hat{\mathbf{n}}$ perpendicular to $\hat{\mathbf{e}}_3$ such that $C(\boldsymbol{\tau}\cdot\hat{\mathbf{n}})=(\boldsymbol{\tau}\cdot\hat{\mathbf{n}})C^\dagger$, and the field solution has the symmetry

$$U_2(\mathbf{x})=C(\boldsymbol{\tau}\cdot\hat{\mathbf{n}})U_2(R_{\hat{\mathbf{n}}}(\pi)\mathbf{x})(\boldsymbol{\tau}\cdot\hat{\mathbf{n}})C^\dagger,$$

with $R_{\hat{\mathbf{n}}}(\pi)$ representing a rotation of π about the $\hat{\mathbf{n}}$ axis [8]. Configurations with $\hat{\mathbf{n}}$'s related by a rotation about the 3 axis are degenerate in energy; so we can restrict

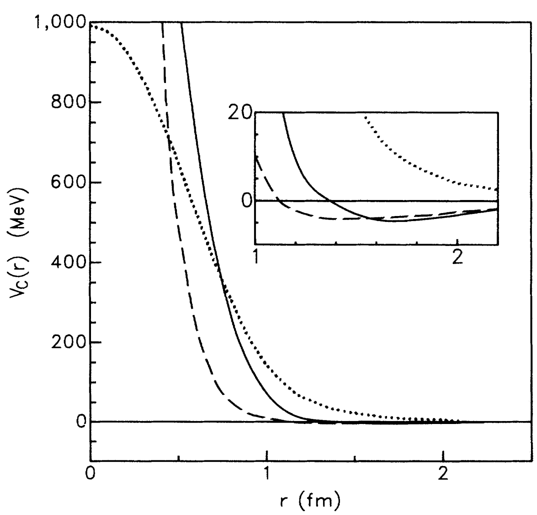

FIG. 1. The central part of the nucleon-nucleon interaction $V_C(r)$ for the present calculations (solid line), the Paris potential [6] (dashed line), and the product approximation [1] (dotted line).

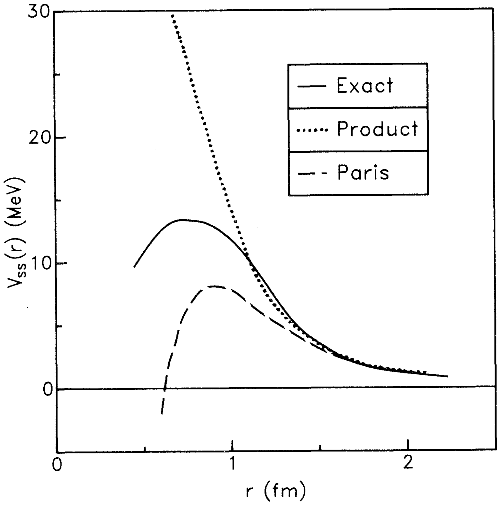

FIG. 2. The spin-spin term of the nucleon-nucleon potential $V_{SS}(r)$, with the curves as in Fig. 1.

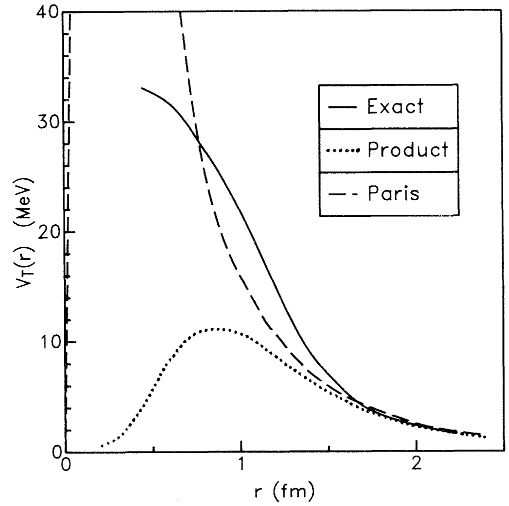

FIG. 3. The tensor term of the nucleon-nucleon potential $V_T(r)$, with the curves as in Fig. 1.

$\hat{\mathbf{n}} = \hat{\mathbf{e}}_1$. We study configurations with the orientations $C = 1$, $i\tau_2$, and $i\tau_3$. For these orientations, the minimum-energy solutions have the additional symmetries

$$U_2(x,y,z) = \tau_1 U_2^\dagger(-x,y,z)\tau_1 = \tau_2 U_2^\dagger(x,-y,z)\tau_2\,,$$

which reduce the calculation to one-eighth of space [9].

The terms in the nucleon-nucleon potential Eq. (5) derived from the calculations for these three orientations are shown in Figs. 1-3. Also shown are the phenomenological Paris potential [6] and the results from the product-approximation calculation of Jackson, Jackson, and Pasquier [1]. We see a better agreement between the exact calculations and the Paris at intermediate and short ranges than was found for the product approximation. In particular, the intermediate-range central attraction, missing in the product approximation, is present here. (We find a similar result for the parameter set of Adkins and Nappi [13]. The core is pushed further out, and the well is slightly deeper.) The behavior at short range for the present potential is quite different from that derived from the product approximation and qualitatively similar to that of the Paris potential. Finally, we remark that a direct quantitative comparison of the Skyrme and Paris potentials is not meaningful at short distances without a knowledge of how the relative Skyrmion mass and moments of inertia depend on r. For small r, these quantities are likely to be quite different from their free values [14]. An r dependence in the mass can be reexpressed as a momentum dependence in the potential. Work in this direction is in progress [15].

To conclude, the present calculations show that even the basic Skyrme model, through a relatively small amount of computational effort, with parameters fixed to the meson sector, and with minimal finite-N_c corrections, predicts a nucleon-nucleon potential which is in qualitative agreement with the phenomenological Paris potential.

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tions, where α_i and β_i are functions of r, one finds $V_{00}=\alpha_1+\beta_1+\alpha_2/4+\alpha_3/4$, $V_{10}=(\alpha_2+\alpha_3)/4$, $V_{11}=(\alpha_2-\alpha_3)/2$, $V_{20}=(\beta_2+\beta_3)/16$, $V_{21}=(\beta_2-\beta_3)/8$, and $V_{22}=(\beta_2+\beta_3+4\beta_5/3)/8$.

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