PHYSICAL REVIEW LETTERS

VOLUME 67 | 25 NOVEMBER 1991 | NUMBER 22

# Dynamical Generation of Long-Range Interactions: Random Levy Flights in the Kinetic Ising and Spherical Models

Birger Bergersen and Zoltán Rácz[a]

*Department of Physics, University of British Columbia, Vancouver, British Columbia, Canada V6T 2A6*

(Received 14 May 1991)

We study nonequilibrium steady states produced by a competition between anomalous diffusion (random Levy-flight exchanges) and local dynamics satisfying detailed balance for the equilibrium state of either the Ising chain or the spherical model. The phase transitions occurring in these systems are investigated via simulations and an exact solution. The results show that the steady-state properties are dominated by effective long-range interactions decaying with distance as $r^{-d-\sigma}$, where $d$ is the spatial dimension while $\sigma$ is the dimension of the Levy flight.

PACS numbers: 05.50.+q, 05.70.Ln, 64.60.Ak

Nonequilibrium steady states are hard to characterize since the inverse problem of finding effective interactions generated by a given dynamics is highly nontrivial. Even if an effective Hamiltonian can be deduced from the stationary distribution function, one usually finds that all the interactions allowed by the symmetry of the system are present, and it is not clear which interactions dominate. In this paper we suggest that the dominant interactions can be extracted directly by the following two-step method: (1) Study the phase transitions in the steady state, and (2) assume that the universality arguments used in equilibrium critical phenomena can be carried over to the nonequilibrium case.

In order to see how the method works, consider, for example, the model we shall be concerned with throughout this paper, namely, the kinetic Ising model in which spin flips satisfying detailed balance for the equilibrium state of the Ising model at temperature $T$ compete with random ($T=\infty$) spin exchanges [1]. If the spin exchanges are between nearest-neighbor sites and their rate is significantly smaller than the rate of spin flips, then the nonequilibrium ordering in the $d=2$ dimensional system is of Ising type [2]. Thus, one may conclude that although the infrequent nearest-neighbor exchanges may generate further neighbor couplings, they do not change the short-range, ferromagnetic nature of the dominant interactions. If the random exchanges have infinite range [3], however, one finds that the ferromagnetic ordering is mean-field-like and, furthermore, the finite-size scaling of the magnetization fluctuations is indistinguishable from that of an equilibrium Ising model with infinite-range interactions. Thus the long-range random exchanges transform the nearest-neighbor ferromagnetic couplings into infinite-range interactions.

Lacking systematic knowledge about the dynamical generation of interactions [4], we wish to pursue the line of argument discussed above by asking the following question: *Are there random exchange processes that generate effective interactions decaying with distance r as a power law* $V_{\mathrm{eff}}(r) \sim r^{-d-\sigma}$? The possibility of studying this question arises since the equilibrium critical behavior of a $d$-dimensional system with ferromagnetic interactions of the form $r^{-d-\sigma}$ is distinct from both the mean-field and the short-range limit, provided that $\sigma$ is from the interval $(d/2, 2-\eta)$, where $\eta$ is the exponent describing critical correlations in the short-range system [5].

In our search for the appropriate random exchange process, we were guided by equivalences between various random walks and ferromagnetic spin models. The nearest-neighbor-interaction spherical model, for example, is known to be equivalent to the ordinary random walk [6]. At the same time, the long-range $[V(r) \sim r^{-d-\sigma}]$ version of this model is equivalent to the Levy flight defined [7] as a random walk in which the step length ($l$) is a random variable with a probability distribution $P(l) \sim l^{-1-\sigma}$ (the parameter $\sigma$ gives the dimension of the random walk for $0<\sigma<2$). Similar equivalences exist in the $n \to 0$ limit of $n$-component spin models, the difference being that both the random walk and the Levy flight are restricted to be node avoiding [8,9].

It is by no means obvious from the above equivalences that Levy flights generate long-range potentials. Note, e.g., that if the Levy-flight exchanges are not random but satisfy detailed balance [10], then the interactions are independent of the exchange process: They are entirely determined by the Hamiltonian in the detailed balance condition. Nevertheless, the above equivalences indicate a possible connection with Levy flights and we were led to consider a $d=1$ kinetic Ising model in which random Levy-flight exchanges ($\sigma=0.75$) compete with spin flips induced by a heat bath at temperature $T$. Our Monte Carlo simulations of this model indicate that the system does undergo a second-order phase transition at a finite spin-flip temperature and the critical fluctuations scale as those of a $d=1$ equilibrium Ising model with long-range interaction of the form $V(r)\sim r^{-1-\sigma}$. As a second example, we consider a generalized time-dependent Landau-Ginzburg model in which a local relaxational mechanism satisfying detailed balance competes with anomalous diffusion representing the random Levy-flight exchanges. This model can be solved exactly in the spherical limit and one can demonstrate explicitly that the dimension of the Levy flight, $\sigma$, determines the exponent in the effective potential $V(r)\sim r^{-\zeta}$ through $\zeta=d+\sigma$.

Let us first consider a kinetic Ising model in which stochastic Ising variables $s_i=\pm 1$ assigned to the sites of a $d=1$ periodic lattice ($i=1,2,\ldots,L$; $s_{i+L}=s_i$) evolve through flips and exchanges of the spins. The flip rate at site $i$ is a version of the Glauber rate [2,11],

$$w_i^{(1)}=\frac{1}{\tau_1(1+v)^2}[1-vs_i(s_{i+1}+s_{i-1})+v^2 s_{i+1}s_{i-1}]\,,$$

where $\tau_1$ just sets the time scale and $v$ controls the temperature of the spin-flip heat bath. If the dynamics consisted of spin flips alone then the system would relax to the equilibrium state of the Ising model at temperature $T$ with the nearest-neighbor coupling $J$ related to $v$ through $v=\tanh(J/kT)$.

The exchanges of spins between sites $i$ and $j$ take place independently of spin flips and their rate is given by

$$w_{i,j}^{(2)}=\frac{A}{\tau_2|i-j|^{1+\sigma}}\,,$$

where $A=2\sum_{i=1}^{\infty}i^{-1-\sigma}$. The above expression for $w_{i,j}^{(2)}$ can be interpreted in the following way. Any given spin moves with a frequency of $1/\tau_2$ and the probability of moving a distance $l$ is proportional to $l^{-1-\sigma}$. Thus the spins execute a Levy flight of dimension $\sigma$. Since the exchanges are independent of energy, one can view this process as being generated by a $T=\infty$ heat bath.

We carried out a Monte Carlo simulation of the above system for $\sigma=0.75$. The choice of $\sigma$ is motivated by the expectation that the exponent of the effective interaction is given by $d+\sigma$. For $d=1$ this interaction produces a phase transition in an equilibrium Ising model [5] only for $\sigma\leq 1$ and the transition is nonclassical only for $\sigma>0.5$. At $\sigma=0.75$ the critical exponents are sufficiently different [12–14] from the mean-field values so as to be easily distinguishable and one is not too close to $\sigma=1$ where strong Kosterlitz-Thouless singularities occur [15].

In the simulations, the frequency of spin-flip attempts was equal to the frequency of spin exchanges ($\tau_1=\tau_2$). Systems of $L=75,150,300,\ldots,4800$ spins were considered and the time evolution of the magnetization, and the energy, was monitored to obtain first a rough estimate of the relaxation time and then the steady-state value of the magnetization fluctuations $\langle M^2\rangle$. The data were then analyzed by assuming that $\langle M^2\rangle$ obeyed finite-size scaling [3,16] near the critical point ($v\approx v_c$):

$$\langle M^2\rangle\approx L^{1+\gamma/\nu}\Phi_\pm(\epsilon L^{1/\nu})\,. \tag{1}$$

Here $\epsilon=|v-v_c|/v_c$ and $\gamma$ and $\nu$ are the critical exponents of the magnetization fluctuations and the correlation length, respectively. The scaling function has two branches, $\Phi_+$ and $\Phi_-$, describing the disordered ($v<v_c$) and ordered ($v>v_c$) phases, respectively. The parameters $v_c$, $\gamma$, and $\nu$ were fitted to give the best collapse of data when $\langle M^2\rangle/L^{1+\gamma/\nu}$ was plotted against $\epsilon L^{1/\nu}$. The open circles in Fig. 1 show the scaling plot for $v_c=0.5945$, $\gamma/\nu=0.65$, and $1/\nu=0.54$. The collapse of data is excellent over 2.5 decades of the scaling variable $\epsilon L^{1/\nu}$. The quality of the scaling declines unambiguously if $v_c$ is changed by more than 0.5% or the critical exponents are altered by more than 5%.

The critical exponent values $\gamma\approx 1.2$ and $\nu\approx 1.9$ are consistent with what is known [5,12–14] about the long-range equilibrium Ising model ($\gamma\approx 1.4$ and $\nu\approx 2.1$ for $\sigma=0.75$), but the uncertainties of the above estimates are rather large. Thus, to make the case of equal exponents more convincing, we have also simulated the equilibrium Ising model with long-range couplings [12] $J_{i,j}\sim|i-j|^{-1.75}$. The solid diamonds in Fig. 1 are $\lambda_2\langle M^2\rangle/L^{1+\gamma/\nu}$ plotted against $\lambda_1\epsilon L^{1/\nu}$ for systems of sizes [17] $L=80,160,\ldots,2560$. We use here the same exponents $\gamma/\nu$ and $1/\nu$ as we did for the Levy-flight system. The fitting parameters are the critical temperature in $\epsilon=|T-T_c|/T_c$ and the scale factors $\lambda_1$ and $\lambda_2$ relating the temperature and magnetization scales in the two systems. Note that on a log-log plot the change in $\lambda_1$ and $\lambda_2$ causes a uniform shift but does not affect the shape of the scaling function. The collapse of the two sets of data shown for $T_c=2.714$, $\lambda_1=0.65$, and $\lambda_2=0.88$ is quite remarkable. Thus we conclude that, at least in $d=1$, the Levy-flight exchanges of dimension $\sigma$ do indeed generate an effective potential of the form $V(r)\sim r^{-1-\sigma}$.

In order to draw more general conclusions, we now turn to a generalized time-dependent Landau-Ginzburg model that is a combination of models A and B, following the classification scheme of critical dynamics [18].

Let an $n$-component order-parameter field, $\hat{S}^i(\mathbf{x},t)$ ($i=1,\ldots,n$), evolve via two processes occurring in parallel. The first one is a local relaxation (model A) satisfying detailed balance at temperature $T$. The second one is an anomalous diffusion process ($T=\infty$ limit of a generalized model B) describing the continuum version of Levy-flight exchanges. (Note that anomalous diffusion is actually not uncommon in nature. An example is the Richardson diffusion [19] in turbulence, which has been argued [20] to be describable by a Levy-flight process.) As has been shown [21], the diffusion operator $Dq^2$ in this case is replaced by $Dq^\sigma$, where $0<\sigma<2$ is the dimension of the Levy flight. Then the equation of motion for the Fourier transform of the field, $S^i_{\mathbf{q}}(t)$, is given by the following Langevin equation:

$$\dot{S}^i_{\mathbf{q}}(t) = -\Gamma_0(r_0+q^2)S^i_{\mathbf{q}}(t) - \Gamma_0 u \sum_{j=1}^{n} \int_{\mathbf{q}'}\int_{\mathbf{q}''} S^j_{\mathbf{q}'}(t) S^j_{\mathbf{q}''}(t) S^i_{\mathbf{q}-\mathbf{q}'-\mathbf{q}''}(t) + \eta^i_{\mathbf{q}}(t) - Dq^\sigma S^i_{\mathbf{q}}(t) + \bar{\eta}^i_{\mathbf{q}}(t)\,. \quad (2)$$

Without the last two terms, Eq. (2) defines model A, which has an equilibrium state with a short-range ferromagnetic Hamiltonian provided that $\eta$ is a Gaussian-Markovian random force with the correlation function

$$\langle \eta^i_{\mathbf{q}}(t)\eta^j_{\mathbf{q}'}(t')\rangle = 2\Gamma_0\delta_{ij}\delta(\mathbf{q}+\mathbf{q}')\delta(t-t')\,.$$

The parameters $\Gamma_0$ and $u$ are constants with $u\sim 1/n$ in the spherical limit $n\to\infty$. The relevant temperature dependence is contained in $r_0\sim a+T$, where $a$ is another constant.

The last two terms in Eq. (2) represent the anomalous diffusion produced by Levy flights. Assuming that $\bar{\eta}$ is again Gaussian white noise with correlations of the form

$$\langle \bar{\eta}^i_{\mathbf{q}}(t)\bar{\eta}^j_{\mathbf{q}'}(t')\rangle = 2Dq^\sigma\delta_{ij}\delta(\mathbf{q}+\mathbf{q}')\delta(t-t')\,,$$

one finds that without the local relaxation process, Eq. (2) yields a structure factor $\langle S^i_{\mathbf{q}}(\infty)S^j_{\mathbf{q}'}(\infty)\rangle = \delta_{ij}\delta(\mathbf{q}+\mathbf{q}')$ of a state in which every configuration is equally probable. In this sense, the Levy flights are generated by contact with a $T=\infty$ heat bath.

The spherical limit ($n\to\infty$) is exactly solvable because fluctuations in $u\sum S^j_{\mathbf{q}}(t)S^j_{\mathbf{q}'}(t)$ may be neglected. This quantity may be replaced by

$$u\sum_{j=1}^{n}\langle S^j_{\mathbf{q}}(t)S^j_{\mathbf{q}'}(t)\rangle = unC(q,t)\delta(\mathbf{q}+\mathbf{q}')\,, \quad (3)$$

where the angular brackets denote averaging over both the initial conditions and the noises $\eta$ and $\bar{\eta}$. Note that the dynamic structure factor $C(q,t)=\langle S^j_{\mathbf{q}}(t)S^j_{-\mathbf{q}}(t)\rangle$ is independent of $j$ if we restrict our studies to the high-temperature phase of the system.

Using (3), the equation of motion (2) becomes linear. Its solution can be used to derive the following self-consistency equation [22,23] for $C(q,t)$:

$$C(q,t) = C(q,0)\exp\left[-\int_0^t \Gamma(q,s)ds\right] + 2(\Gamma_0+Dq^\sigma)\int_0^t dt' \exp\left[-\int_0^{t'}\Gamma(q,s)ds\right]. \quad (4)$$

Here, $C(q,0)$ is the initial structure factor, and $\Gamma(q,t)$ is given by

$$\Gamma(q,t) = 2\Gamma_0[r_0+q^2+unC(q,t)] + 2Dq^\sigma\,. \quad (5)$$

One can see from (5) that the effective relaxation rate $\Gamma(q,t)$ is increased by the addition of Levy flights. Since $C(q,t)$ would reach a long-time limit $C(q)$ even without the $2Dq^\sigma$ term in $\Gamma(q,t)$, it will do so in the nonequilibrium case as well. Then the equation for $C(q)$ obtained from the $t\to\infty$ limit of (4) is

$$C(q) = \frac{\Gamma_0+Dq^\sigma}{\Gamma_0(r_0+q^2+unS)+Dq^\sigma}\,, \quad (6)$$

where $S=\int d\mathbf{q}\,C(q)$. The long-wavelength instabilities are determined by the $q\to 0$ form of $C(q)$, which for $0<\sigma<2$ can be written as

$$C(q) \approx (r_0+\lambda q^\sigma+unS)^{-1}\,, \quad (7)$$

with $\lambda = D/\Gamma_0$. This form coincides with the long-wavelength limit of the equilibrium structure factor of a spherical model in which the interactions decay with distance as $r^{-d-\sigma}$. Consequently, both the self-consistency equation for $r=r_0+unS$ and the critical behavior that follows from it are identical to that of the equilibrium long-range model. Thus we can conclude that the critical properties of the nonequilibrium steady state are dom-

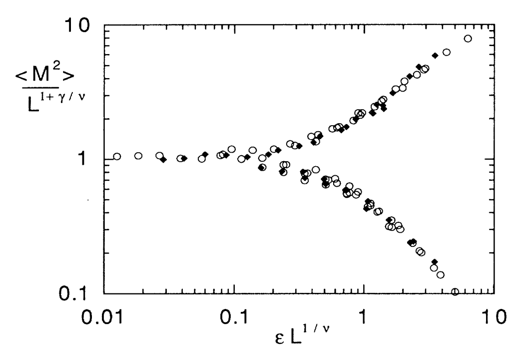


FIG. 1. Finite-size scaling of the magnetization fluctuations $\langle M^2\rangle$ in the steady state of the flip-and-Levy-flight model (○), and in the equilibrium state of the long-range-interaction ($J_{i,j}\sim|i-j|^{-1.75}$) Ising model (◆). The critical exponents are the same ($\gamma/\nu=0.65$ and $1/\nu=0.54$) for both sets of data. As explained in the text, the deviations from the critical point in the two systems are given by $\epsilon=|v_c-v|/v_c=0.65|T-T_c|/T_c$, and the scale of $\langle M^2\rangle$ on the plot of the Ising data differs from that of the Levy-flight data by a factor of $\lambda_2=0.88$.

inated by an effective long-range potential proportional to $r^{-d-\sigma}$.

The above conclusion should be valid for finite $n$ as well. Looking at Eq. (2), one can see that the correlations in the effective noise ($\eta_{\text{eff}}=\eta+\bar{\eta}$) have an amplitude $2(\Gamma_0+Dq^\sigma)$. One expects that the $Dq^\sigma$ term can be neglected in the long-wavelength limit and thus that the noise $\bar{\eta}$ in the Levy-flight exchanges can be omitted. Without $\bar{\eta}$, however, the system described by Eq. (2) satisfies detailed balance and has an effective Hamiltonian which, apart from the usual short-range interaction pieces, contains the expected long-range part $\lambda\int d\mathbf{q}\, q^\sigma \sum_i S^i(\mathbf{q})S^i(-\mathbf{q})$. The missing link in the above argument is a proof that the noise in the Levy flight is irrelevant. This problem can, in principle, be investigated by field-theoretic methods but the lack of a fluctuation-dissipation theorem makes the calculations rather involved.

We are grateful to H. W. Diehl, M. Plischke, and H.-J. Xu for helpful discussions. This work was supported by the Natural Sciences and Engineering Research Council of Canada.

[a]Permanent address: Institute for Theoretical Physics, Eötvös University, 1088 Budapest, Puskin u. 5-7, Hungary.

[1] A. DeMasi, P. A. Ferrari, and J. L. Lebowitz, Phys. Rev. Lett. **55**, 1947 (1985); J. Stat. Phys. **44**, 589 (1986).

[2] J. M. Gonzalez-Miranda, P. L. Garrido, J. Marro, and J. L. Lebowitz, Phys. Rev. Lett. **59**, 1934 (1987); J-S. Wang and J. L. Lebowitz, J. Stat. Phys. **51**, 893 (1988).

[3] M. Droz, Z. Rácz, and P. Tartaglia, Phys. Rev. A **41**, 6621 (1990).

[4] P. L. Garrido and J. Marro, Phys. Rev. Lett. **62**, 1929 (1989).

[5] M. E. Fisher, S. K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972); J. Sak, Phys. Rev. B **8**, 281 (1973).

[6] G. S. Joyce, in *Phase Transitions and Critical Phenomena,* edited by C. Domb and M. S. Green (Academic, London, 1972), Vol. 2, p. 375.

[7] B. B. Mandelbrot, *Fractals: Form, Chance, and Dimension* (Freeman, San Francisco, 1977).

[8] M. Daoud *et al.*, Macromolecules **8**, 804 (1975).

[9] J. W. Halley and H. Nakanishi, Phys. Rev. Lett. **55**, 551 (1985).

[10] H. Hayakawa and F. Family, Physica (Amsterdam) **166A**, 408 (1990).

[11] R. J. Glauber, J. Math. Phys. **4**, 294 (1963).

[12] J. F. Nagle and J. C. Bonner, J. Phys. C **3**, 352 (1970).

[13] J. L. Monroe, R. Lucente, and J. P. Hourlland, J. Phys. A **23**, 2555 (1990).

[14] K. Uzelac and Z. Glumac, J. Phys. A **21**, L421 (1988).

[15] J. L. Cardy, J. Phys. A **17**, L385 (1984).

[16] A collection of reviews: *Finite Size Scaling and Numerical Simulation of Statistical Systems,* edited by V. Privman (World Scientific, Singapore, 1990).

[17] The long-range nature of the interactions makes the simulations rather time consuming. It took about half a year of CPU time on three Sparc1 workstations to accumulate the data shown in Fig. 1.

[18] P. C. Hohenberg and B. I. Halperin, Rev. Mod. Phys. **49**, 435 (1977).

[19] L. F. Richardson, Proc. R. Soc. London A **110**, 709 (1926).

[20] M. F. Schlesinger, B.J. West, and J. Klafter, Phys. Rev. Lett. **58**, 1100 (1987).

[21] B. D. Houges, M. F. Schlesinger, and E. Montroll, Proc. Natl. Acad. Sci. (U.S.A.) **78**, 3287 (1981); J. Stat. Phys. **28**, 111 (1982).

[22] Z. Rácz and T. Tél, Phys. Lett. **60A**, 3 (1977).

[23] Note that the equation for $C(q,t)$ can be obtained by field-theoretic methods by iterating Eq. (2). In the $n\to\infty$ limit one has to keep track of the closed loops since $u\sim n^{-1}$ and each loop generates a power of $n$.