Violations of a Simple Inequality for Classical Fields

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It has been shown that two correlated photons incident upon two distant interferometers can give a coincidence counting rate that depends nonlocally on the sum of the phases of the two interferometers. It is shown here that the results of existing experiments violate a simple inequality that must be satisfied by any classical or semiclassical field theory. The inequality provides a graphic illustration of the lack of objective realism of the electric field.

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It has been shown [1,2] that two-photon interferometer experiments can violate Bell's inequality [3] and a number of experiments [4–7] have demonstrated effects of that kind. Current experiments based upon the two-photon interferometer of Ref. [1] have not, however, violated Bell's inequality due to the limited visibility (50%) of the interference fringes that results when the resolving time of the photon detectors and electronics is not sufficiently fast. It will be shown here that those experiments do violate a surprisingly simple inequality that must be satisfied by any classical or semiclassical field theory. The inequality follows directly from the assumption that the classical field has some well-defined value and thus illustrates the lack of objective realism exhibited by the quantum-mechanical field.

The experiments of interest [4,5,7] are outlined in Fig. 1. Two coincident photons are emitted by parametric down-conversion [8] and travel in different directions toward two identical interferometers. Each interferometer contains a shorter and a longer path, and the difference ΔT in transit times over the two paths is taken to be much larger than the coherence time of the photons. Nevertheless, interference between the quantum-mechanical amplitudes for the photons to have both traveled the shorter paths or the longer paths produces a modulation in the coincidence counting rate R_c given [1] by

$$R_c = \tfrac{1}{4} R_{c0} \cos^2\left(\frac{\theta_1+\theta_2+\omega_0\Delta T}{2}\right). \tag{1}$$

Here R_{c0} is the coincidence rate with the beam splitters removed, θ_1 and θ_2 are phase shifts introduced into the two longer paths, and ω_0 is the frequency of the pump laser. Equation (1) violates Bell's inequality but is only valid if the resolution of the coincidence measurements is better than ΔT. The maximum visibility is 50% for time resolutions much worse than ΔT.

There has been considerable debate as to whether or not the experiments with visibilities of 50% or less are nevertheless inconsistent with any semiclassical field theory. Ou and Mandel [9] have suggested that that is the case but counterexamples to their argument have been given by Carmichael [10] and by Chiao and Kwiat [11]. Although their semiclassical models are able to reproduce the modulation in the coincidence rate, they are not able to represent the fact that the photons are known from other experiments [12] to be coincident to within a time interval much smaller than ΔT. That provides the physical basis for the inequalities derived below.

The desired inequality is a generalization of Cauchy's inequality [13], according to which

$$2ab \le a^2+b^2, \tag{2}$$

where a and b are real numbers. When a and b are complex we still have

$$|ab| = |a||b| \le (|a|^2+|b|^2)/2. \tag{3}$$

The modulation of the coincidence rate in a classical treatment of the two-photon interferometer experiments will be found to be proportional to the quantity Q defined by

$$Q \equiv \langle |E_1^*(t)E_2^*(t)E_2(t-\Delta T)E_1(t-\Delta T)|\rangle. \tag{4}$$

Here E_1 and E_2 refer to the fields at the positions of detectors 1 and 2 (which will be assumed to be equidistant from the source) with the beam splitters removed and the angular brackets denote an average over a long time interval. The desired inequality can be obtained by choosing

$$a = E_1^*(t)E_2(t-\Delta T), \tag{5}$$

$$b = E_2^*(t)E_1(t-\Delta T). \tag{6}$$

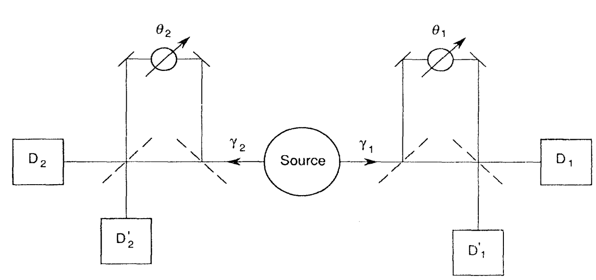

FIG. 1. Two-photon interferometer.

Inserting Eqs. (5) and (6) into Eq. (3) immediately gives

$$\langle|E_1^*(t)E_2^*(t)E_2(t-\Delta T)E_1(t-\Delta T)|\rangle$$
$$\le\langle E_1^*(t)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t)\rangle/2$$
$$+\langle E_2^*(t)E_1^*(t-\Delta T)E_1(t-\Delta T)E_2(t)\rangle/2\,. \quad (7)$$

Somewhat analogous inequalities based instead on the Schwarz or Cauchy-Schwarz inequality have been previously derived by Glauber, Titulaer, and Clauser [14].

The physical significance of the above inequality can be illustrated by the extreme situation shown in Fig. 2, in which both fields $E_1(t)$ and $E_2(t)$ correspond to narrow pulses emitted at the same time. If E_1 is evaluated at time t and E_2 is evaluated at time $t\pm\Delta T$, as illustrated by the arrows in the figure, then one or the other of the fields must be zero and their product vanishes. The right-hand side of Eq. (7) is then zero, which requires that the left-hand side also vanish. Although this inequality may seem trivial in nature, it is a consequence of the fact that the classical fields are well-defined (complex) numbers; the inequality is violated by quantum fields, as will be discussed below.

The inequality of Eq. (7) will now be used to set a limit on the amount of modulation that can occur in a classical treatment of the two-photon interferometer experiments. Once again, let $E_1(t)$ be the classical field that would arrive at detector 1 in the absence of the two beam splitters and assume for the moment that the half-width w of the coincidence window is negligibly small. The corresponding coincidence rate as a function of the time offset τ is

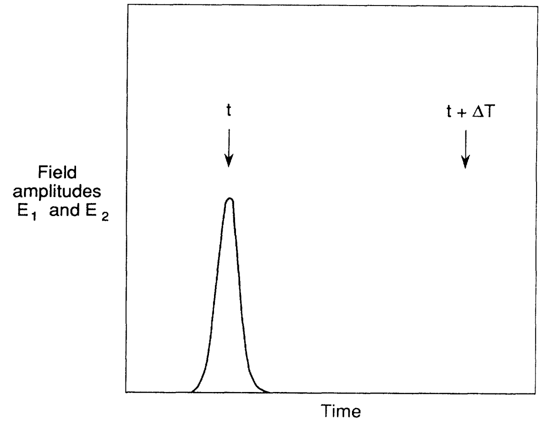

FIG. 2. A pair of classical coincident pulses.

then

$$R_{c0}(\tau)=\eta\langle I_1(t)I_2(t+\tau)\rangle$$
$$=\eta\langle E_1^*(t)E_2^*(t+\tau)E_2(t+\tau)E_1(t)\rangle\,, \quad (8)$$

where I_1 and I_2 are the intensities of the two beams and the constant η is related to the detection efficiencies and w. With the insertion of the two beam splitters, the total electric field $E_{T1}(t)$ at detector 1 becomes

$$E_{T1}=\tfrac{1}{2}[E_1(t)+e^{i\theta_1}E_1(t-\Delta T)]\,. \quad (9)$$

A similar expression exists for the total field at detector 2 and the classical coincidence rate R_c with the beam splitters inserted and $\tau=0$ is given by

$$R_c=\tfrac{1}{16}\eta\langle|[E_1(t)+e^{i\theta_1}E_1(t-\Delta T)][E_2(t)+e^{i\theta_2}E_2(t-\Delta T)]|^2\rangle\,. \quad (10)$$

Multiplying out all the factors in Eq. (10) gives a total of sixteen terms:

$$\begin{aligned}R_c=\tfrac{1}{16}\eta\langle &E_1^*(t)E_2^*(t)E_1(t)E_2(t)+e^{i\theta_2}E_1^*(t)E_2^*(t)E_1(t)E_2(t-\Delta T)\\
&+e^{i\theta_1}E_1^*(t)E_2^*(t)E_1(t-\Delta T)E_2(t)+e^{i[\theta_1+\theta_2]}E_1^*(t)E_2^*(t)E_1(t-\Delta T)E_2(t-\Delta T)\\
&+e^{-i\theta_2}E_1^*(t)E_2^*(t-\Delta T)E_1(t)E_2(t)+E_1^*(t)E_2^*(t-\Delta T)E_1(t)E_2(t-\Delta T)\\
&+e^{i[\theta_1-\theta_2]}E_1^*(t)E_2^*(t-\Delta T)E_1(t-\Delta T)E_2(t)+e^{i\theta_1}E_1^*(t)E_2^*(t-\Delta T)E_1(t-\Delta T)E_2(t-\Delta T)\\
&+e^{-i\theta_1}E_1^*(t-\Delta T)E_2^*(t)E_1(t)E_2(t)+e^{i[\theta_2-\theta_1]}E_1^*(t-\Delta T)E_2^*(t)E_1(t)E_2(t-\Delta T)\\
&+E_1^*(t-\Delta T)E_2^*(t)E_1(t-\Delta T)E_2(t)+e^{i\theta_2}E_1^*(t-\Delta T)E_2^*(t)E_1(t-\Delta T)E_2(t-\Delta T)\\
&+e^{i[-\theta_1-\theta_2]}E_1^*(t-\Delta T)E_2^*(t-\Delta T)E_1(t)E_2(t)+e^{-i\theta_1}E_1^*(t-\Delta T)E_2^*(t-\Delta T)E_1(t)E_2(t-\Delta T)\\
&+e^{-i\theta_2}E_1^*(t-\Delta T)E_2^*(t-\Delta T)E_1(t-\Delta T)E_2(t)+E_1^*(t-\Delta T)E_2^*(t-\Delta T)E_1(t-\Delta T)E_2(t-\Delta T)\rangle\,. \quad (11)\end{aligned}$$

As suggested by Eq. (1), the experiments can be performed in such a way as to measure the averaged coincidence rate as a function of $\theta_T=\theta_1+\theta_2$:

$$\bar{R}_c(\theta_T)=\frac{1}{2\pi}\int_0^{2\pi}d\theta_1\int_0^{2\pi}d\theta_2\,R_c(\theta_1,\theta_2)\,\delta(\theta_1+\theta_2-\theta_T)\,. \quad (12)$$

The averages over θ_1 and θ_2 were explicitly performed in one of the experiments [7], while thermal drifts had the effect of averaging over the phases in the remaining experiments, since the individual phases were not directly measured and had essentially random values from one run to the next. In any event, the terms in Eq. (11) with phase factors of $\exp(i\theta_1)$, $\exp(i\theta_2)$, $\exp[i(\theta_1-\theta_2)]$, etc., average to zero, leaving only those terms with no phase dependence or a dependence on $\theta_1+\theta_2$. The remaining terms can be written as

$$\bar{R}_c=\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t)E_2(t)E_1(t)\rangle+\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t)\rangle$$
$$+\tfrac{1}{16}\eta[e^{i\theta_T}\langle E_1^*(t)E_2^*(t)E_2(t-\Delta T)E_1(t-\Delta T)\rangle+\text{c.c.}]\,, \tag{13}$$

where the average over a long time interval ensures that

$$\langle E_1^*(t-\Delta T)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t-\Delta T)\rangle=\langle E_1^*(t)E_2^*(t)E_2(t)E_1(t)\rangle \tag{14}$$

and the symmetry of the two beams gives

$$\langle E_1^*(t)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t)\rangle=\langle E_2^*(t)E_1^*(t-\Delta T)E_1(t-\Delta T)E_2(t)\rangle\,. \tag{15}$$

The assumption inherent in Eq. (15) is not essential and can be avoided by simply replacing $R_{c0}(\Delta T)$ with $[R_{c0}(\Delta T)+R_{c0}(-\Delta T)]/2$ in what follows.

The maximum and minimum coincidence rates from Eq. (13) satisfy

$$R_{\max}\le\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t)E_2(t)E_1(t)\rangle+\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t)\rangle$$
$$+\tfrac{1}{8}\eta\langle|E_1^*(t)E_2^*(t)E_2(t-\Delta T)E_1(t-\Delta T)|\rangle\,, \tag{16}$$

$$R_{\min}\ge\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t)E_2(t)E_1(t)\rangle+\tfrac{1}{8}\eta\langle E_1^*(t)E_2^*(t-\Delta T)E_2(t-\Delta T)E_1(t)\rangle$$
$$-\tfrac{1}{8}\eta\langle|E_1^*(t)E_2^*(t)E_2(t-\Delta T)E_1(t-\Delta T)|\rangle\,. \tag{17}$$

The visibility is defined as usual by

$$v=\frac{R_{\max}-R_{\min}}{R_{\max}+R_{\min}}\,. \tag{18}$$

Using the inequality of Eq. (7) and expressing the right-hand side in terms of $R_{c0}(\Delta T)$ gives

$$v\le\frac{R_{c0}(\Delta T)}{R_{c0}(0)+R_{c0}(\Delta T)}\,. \tag{19}$$

Equation (19) limits the visibility that can occur in any classical field theory and gives zero modulation for the case in which the fields correspond to coincident pulses.

If the experiments are performed using detectors with limited time responses and large coincidence windows, as is often the case, then the above inequality can be generalized [15] to

$$v\le\frac{\int_{\Delta T/2}^{\infty}R_{c0}(\tau)d\tau+\frac{1}{2}\int_{\Delta T/2}^{3\Delta T/2}R_{c0}(\tau)d\tau}{2\int_0^{\infty}R_{c0}(\tau)d\tau}\,. \tag{20}$$

Here R_{c0} is again the coincidence rate that would be obtained using detectors with a negligible time response and a negligibly small window.

Earlier experiments [12] have already shown that the down-converted photons are coincident to within a time interval much less than the value of ΔT in at least two [5,7] of the two-photon interferometer experiments, in which case the inequalities of Eq. (19) or (20) show that there is no classical or semiclassical field theory consistent with all of the available observations [16].

In the classical models suggested by Carmichael [10] and by Chiao and Kwiat [11], the fields E_1 and E_2 have well-defined frequencies that sum to the pump-laser frequency for a time interval larger than ΔT or the time resolution of the coincidence circuits. In that case the coincidence rate of Eq. (13) simplifies to

$$\bar{R}_c=\tfrac{1}{4}R_{c0}\left[\cos^2\left(\frac{\theta_T+\omega_0\Delta T}{2}\right)+\frac{1}{2}\right]\,. \tag{21}$$

This differs from the quantum-mechanical result by the additional factor of $\frac{1}{2}$ and corresponds to a visibility of 50%. Such models cannot simultaneously localize the fields into coincident pulses whose widths are less than ΔT, however. Any classical model that does would have the visibility reduced accordingly as required by the inequalities of Eq. (19) or (20).

In quantum optics the intensity operator is given by $I(t)=E^-(t)E^+(t)$, where E^+ and E^- are the positive- and negative-frequency components of the electric-field operator [13]. As a result, the quantum-mechanical equivalent of Eq. (7) would be

$$|\langle E_1^-(t)E_2^-(t)E_2^+(t-\Delta T)E_1^+(t-\Delta T)\rangle|\le\langle E_1^-(t)E_2^-(t-\Delta T)E_2^+(t-\Delta T)E_1^+(t)\rangle/2$$
$$+\langle E_2^-(t)E_1^-(t-\Delta T)E_1^+(t-\Delta T)E_2^+(t)\rangle/2\,. \tag{22}$$

It has already been noted [1] that in experiments of this kind the coincidence of the photons requires

$$E_1^+(t)E_2^+(t\pm\Delta T)=0\,, \tag{23}$$

while conservation of energy in the parametric down-conversion process requires that

$$E_1^+(t-\Delta T)E_2^+(t-\Delta T)=e^{i(\omega_1+\omega_2)\Delta T}E_1^+(t)E_2^+(t)\,, \tag{24}$$

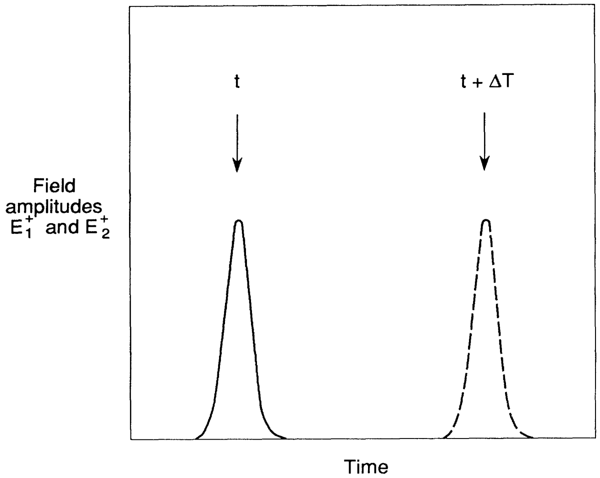

FIG. 3. Quantum-mechanical field corresponding to an entangled pair of coincident photons, with a superposition of times at which the pair may have been emitted.

where the sum of the two photon frequencies ω_1 and ω_2 is equal to ω_0. [Equation (24) is only valid when ΔT is small compared to the pump-laser coherence time.] Inserting Eq. (23) into the right-hand side of Eq. (22) gives zero, whereas inserting Eq. (24) into the left-hand side gives $\langle E_1^-(t)E_2^-(t)E_2^+(t)E_1^+(t)\rangle$, which is the product of the individual beam intensities and a nonzero quantity. Thus the inequality is violated in quantum optics.

The quantum-mechanical situation is illustrated in Fig. 3. The field corresponds to an entangled state in which there is a superposition of times at which the pair of photons may have been emitted, as indicated by the existence of both the solid and dotted curves. Although the product of E_1^+ and E_2^+ at two different times is zero, that does not imply that the left-hand side of Eq. (22) must vanish. Equations (23) and (24) would be logically inconsistent if the fields were well-defined complex numbers and the violation of this inequality provides a graphic demonstration of the lack of objective realism of the electric field.

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