Glassy Dynamics of Two-Dimensional Vortex Glasses, Charge-Density Waves, and Surfaces of Disordered Crystals

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The low-temperature phase of a model of pinned, two-dimensional flux lines is analytically shown to be glassy. Typical energy barriers L diverge as $(\ln L)^{1/2}$ as the length scale $L\to\infty$. This implies a voltage-current relation of the form $V=C_1 I\exp\{-C_2[\ln(I_0/I)]^{1/2}\}$. The growth velocity V_G of the surface of a disordered crystal is given by $V_G=C_3\Delta\mu\exp\{-C_4[\ln(\Delta\mu_c/\Delta\mu)]^{1/2}\}$, where $\Delta\mu$ is the crystal-liquid chemical-potential difference. Similar results hold for 2D charge-density waves, if dislocations in the charge-density wave are ignored.

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The Abrikosov flux-lattice state in a perfectly clean material is *not* superconducting [1]; it has a nonzero resistance R at small current I due to motion of the flux lines. R remains finite (though very small) in dirty superconductors undergoing [2] activated flux creep. A new phase of pinned flux lattices called the "vortex glass" which *is* a true superconductor has been proposed [3–5], and possibly seen experimentally [6].

Pinning destroys both translational [7] and orientational [8] long-ranged order for spatial dimensions $d<4$. But the vortex glass has [3–5] "spin glass" order, in which the flux lines lock into some (random) low free-energy configuration. As in spin glasses [9], excitations of linear spatial extent L out of this state are assumed to typically cost an energy $E_T(L)\propto L^{\theta}$ as $L\to\infty$, with θ (>0) universal. The assumption that $E_T(L\to\infty)\to\infty$ implies [10] that the resistance $R\equiv\lim_{I\to 0}dV/dI=0$. Numerical [11] and experimental [6] evidence favors this scenario, but convincing *analytic* arguments that $E_T(L\to\infty)\to\infty$ have been lacking [12].

This paper provides such a demonstration for a model of randomly pinned flux lines in $d=2$, introduced in Ref. [3]. This model describes [13] a thin film of superconductor sandwiched between two bulk superconductors in an applied magnetic field H parallel to the film that satisfies $H_{c1}^{\text{film}}<H<H_{c1}^{\text{bulk}}$, so that flux lines penetrate *only* the film. Specifically, I find that in the low-temperature (glass) phase of this model

$$E_T(L)=A(T)[\ln(L/a)]^{1/2} \tag{1}$$

as $L\to\infty$. The temperature-dependent prefactor in (1) vanishes linearly as $T\to T_G^-$:

$$A(T)=C(T_G-T)\,, \tag{2}$$

where $C=O(k_B)$ is a nonuniversal constant, and T_G the glass transition temperature.

Assuming that the energy barriers that the system encounters while growing are of order of the typical barriers given by (1) leads to a voltage-current law of the form

$$V=R_0 I\exp\left\{-\alpha\frac{A(T)}{T}\left[\ln\left(\frac{I_0}{I}\right)\right]^{1/2}\right\}, \tag{3}$$

where R_0, α, and I_0 are current independent, and $\alpha=O(1)$. In the "film sandwich" geometry described earlier, both the current density and the electric field are *normal* to the film. This form (3) implies that $R\equiv\lim_{I\to 0}dV/dI=0$, and leads to persistent current decays with time in remnant magnetization experiments of the form

$$I(t)=C_3\exp(-C_4\ln^2 t)\propto t^{-C_4\ln t}. \tag{4}$$

Equation (3) can be transcribed into the growth law for the surface of a bulk-disordered crystal by replacing V with the growth velocity V_G of the surface, and I with $\Delta\mu$, the chemical-potential difference between solid and liquid [14]. This growth is much faster as $\Delta\mu\to 0$ than that predicted [15] for screw-dislocation-catalyzed growth $[V_G\propto(\Delta\mu)^2]$. The voltage-current relation for a randomly pinned 2D charge-density-wave [16] (CDW) system *without dislocations* [17] has the form (3) with V and I interchanged.

I will now sketch the derivation of these results, beginning with a review of Ref. [3]. The model describes a set of thermally meandering, on average parallel, flux lines with a finite line length per unit area ρ_L in the plane, and a finite line tension ε, moving in a random potential. The lines interact via some short-ranged, repulsive potential.

This problem can be mapped [3] onto the problem of surface fluctuations of a bulk-disordered crystal [18]. That problem is described by the model [18,19]

$$H_r\equiv\frac{H}{k_B T}=\int d^2r\{\tfrac{1}{2}\tilde{K}|\nabla h|^2-V\cos[2\pi h(\mathbf{r})+\phi(\mathbf{r})]\}\,, \tag{5}$$

where $h(\mathbf{r})$ gives the local height of the crystal surface above a reference crystallographic plane, and $\phi(\mathbf{r})$ is a quenched random variable that is uniformly distributed between 0 and 2π at the pinning sites, and 0 elsewhere. Tilted boundary conditions $h(x=0,y)=0$, $h(x=L,y)=\theta L$, with $\theta=\rho_L$, are imposed.

To see the connection between this model and the flux line system, first consider its ground state in the absence of disorder [$\phi(\mathbf{r})=0$ everywhere]. A straightforward minimization shows that this ground state is a set of flat

terraces in which $h(\mathbf{r})$ = integer separated by uniformly spaced steps parallel to the y axis of width $w \sim (\tilde{K}/V)^{1/2}$ and step spacing $l = 1/\theta$. If we choose $\tilde{K}$ and V such that $w = \lambda$ (the London penetration depth), we can identify the steps on this surface with the flux lines in the original problem, because the structure, energetics, and density of the two are essentially the same. In particular, the ground states (uniformly spaced lines) are the same in both cases.

Now consider the effect of the random phase $\phi(\mathbf{r})$. If the disorder is weak (i.e., the density of pinning sites is low) the local structure of the surface will remain, over most of the surface, one of terraces separated by well-defined steps. On the pinning sites, the cosine term in (5) is forced away from its energetically favored value of 1 if a terrace (region of h = integer) sits on $\mathbf{r}_P$. However, if a step passes through $\mathbf{r}_P$, one can make a small adjustment in the step's position such that $h(\mathbf{r}_P)$ = integer $-\phi(\mathbf{r}_P)/2\pi$ and thereby make the cosine take on its optimal value there as well. Thus, the pinning sites attract steps, in precisely the same way that the pinning sites in the original flux line problem attract the flux lines. The attraction is short ranged (its range of order the step width w), and so this model is in the same universality class as the original flux line problem.

In this surface model, the growth velocity V_G is equal to the rate at which steps pass a given point. In the superconductor, the *voltage* V is proportional to the rate at which *flux lines* pass a given point. Thus, V_G(surface model) $\propto V$(superconductor).

The model (5) is a Gaussian approximation to a random field X-Y model [20,21]. The ensemble-averaged free energy $\langle F \rangle = -k_B T \langle \ln Z(\{\phi(\mathbf{r})\}) \rangle$, where the angular brackets denote an average over the quenched random variables $\{\phi(\mathbf{r})\}$, determines its thermodynamics. The quenched average $\langle \ln Z \rangle$ can be evaluated by replicas; i.e., using the identity

$$F = -k_B T \lim_{n \to 0} \frac{Z^n - 1}{n} \tag{6}$$

to reduce $\langle F \rangle$ to the partition function for a system with n identical replicas of h,

$$\langle Z^n \rangle = \int \left[\prod_{\alpha=1}^{n} Dh_\alpha \right] \left\langle \exp\left[-\sum_{\alpha'=1}^{n} H\{h_{\alpha'}, \phi\} \right] \right\rangle . \tag{7}$$

Performing the quenched average over ϕ perturbatively in V yields an effective Hamiltonian for the fields h_α alone of the form [21,22]

$$H_r = \int d^2 r \left[\tfrac{1}{2} \tilde{K} \sum_\alpha |\nabla h_\alpha|^2 + \tfrac{1}{2} K_2 \sum_{\alpha,\beta=1}^{n} |\nabla (h_\alpha - h_\beta)|^2 - U \sum_{\alpha,\beta=1}^{n} \cos[2\pi (h_\alpha - h_\beta)] \right] , \tag{8}$$

where $K_2 \equiv 0$ initially, but becomes nonzero under renormalization, and $U \equiv V^2$.

Standard renormalization-group techniques imply [20, 21] that the model undergoes an equilibrium phase transition at $\tilde{K} = \pi$. This is exact, since $\tilde{K}$ is unrenormalized, to all orders, in this model [21]. For temperatures above this transition ($\tilde{K} < \pi$), U renormalizes to zero and K_2 renormalizes to a finite value. For temperatures *below* this transition,

$$U(l \to \infty) \equiv U_* = \frac{2 - 2\pi/\tilde{K}}{C_2}, \quad C_2 = \text{const}, \tag{9}$$

$$K_2(l) = K_a + C_1 U_*^2 l, \quad C_1, K_a \text{ finite constants}, \tag{10}$$

where e^l is the renormalization-group rescaling factor.

This concludes the review of Ref. [3]. This paper will demonstrate that the divergence of K_2 leads to the $[\ln(L)]^{1/2}$ divergence of energy barriers [Eq. (1)].

Thermally activated growth is dominated by those paths through configuration space which encounter the smallest energy barriers; here that path is the nucleation of a "unit plateau," i.e., a finite region within which $h(\mathbf{r}) \to h(\mathbf{r}) + 1$. Such an excitation is low in energy because the free energy (5), being periodic in h, is unchanged by the creation of this plateau, except at its boundary, which is a step.

I will assume that the free energy of this step scales the same way with its spatial extent L as the free energy of a single step of unit height forced into an $L \times L$ system by tilted boundary conditions $h(x=0) = 0$, $h(x=L) = \theta L$, with $\theta = 1/L$. (Here I have already made a change of variables to shift away the tilt $\theta = \rho_L$ imposed by the finite line density.) The interest is in the *change* in the free energy, i.e., $\Delta F(\theta) \equiv F(\theta) - F(0)$. In this disordered system, $\Delta F(\theta)$ is itself a random variable. Its mean is [18]

$$\langle \Delta F(\theta) \rangle = \tfrac{1}{2} \tilde{K} L^2 \theta^2 k_B T \tag{11}$$

exactly, both above and below T_G. Since $\tilde{K}$ is unrenormalized, there is *no* signature of the glass transition in $\langle \Delta F(\theta) \rangle$. Equation (11) can be derived as follows: Upon replicating, the tilted boundary conditions are reproduced in each replica; i.e., $h_\alpha(x=0) = 0$, $h_\alpha(x=L) = \theta L$, for all α. One can now make a change of variables, $h'_\alpha(\mathbf{r}) = h_\alpha(\mathbf{r}) - \theta x$ for all α; since the K_2 and U terms in H_r depend only on *differences* $h_\alpha - h_\beta$, both are unaltered by this change. In fact, only the $\tilde{K}$ term is altered (by $\frac{1}{2} \tilde{K} \theta^2 L^2$); this leads directly to [18]

$$\langle Z^n(\theta) \rangle = \langle Z^n(0) \rangle e^{-n\tilde{K} L^2 \theta^2/2}, \tag{12}$$

where $Z^n(\theta)$ is the partition function for n replicas, all tilted by θ. This (12) implies (11).

Taking $\theta = 1/L$ in (11) reveals that the mean free energy of a single step is *finite*, as $L \to \infty$, in both phases. However, its *fluctuations* diverge, as will now be shown.

The mean-squared fluctuations in $\Delta F(\theta)$ can also be obtained using (6):

$$\langle [F(\theta)-F(0)]^2\rangle = \left\langle \left[\lim_{n\to 0} \left[\frac{Z^n(\theta)-Z^n(0)}{n} \right] \right]^2 \right\rangle (k_BT)^2$$

$$= \lim_{n,n'\to 0} \frac{(k_BT)^2}{nn'} [\langle Z^{n+n'}(\theta)\rangle + \langle Z^{n+n'}(0)\rangle - \langle Z^n(\theta)Z^{n'}(0)\rangle - \langle Z^n(0)Z^{n'}(\theta)\rangle] . \tag{13}$$

Consider the terms $\langle Z^n(\theta)Z^{n'}(0)\rangle$ and $\langle Z^n(0)Z^{n'}(\theta)\rangle$. The first (second) of these can be thought of as the partition function for an $n+n'$ replica system, with n (n') of the replicas tilted by θ and n' (n) of them untilted. Thus, different replicas are now tilted by *different* angles, and hence the K_2 and U terms in the free energy can come into play.

At high temperatures ($\tilde{K} < \pi$), since U renormalizes to zero, $\langle Z^n(\theta)Z^{n'}(0)\rangle$ can be calculated from the Hamiltonian (8) with $U=0$. Since the desired average is the partition function for $n+n'$ replicas, n tilted and n' not, I will shift $h_a \to h_a - \theta x$ for the n tilted replicas alone. This changes only the $\tilde{K}$ terms and those pieces of the K_2 term with α one of the n indices that is tilted and β one of the n' that is not. Thus

$$\langle Z^n(\theta)Z^{n'}(0)\rangle = \langle Z^{n+n'}(0)\rangle \exp\left(-\frac{n}{2}\tilde{K}\theta^2L^2 - \tfrac{1}{2}K_2 \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n'} \theta^2L^2 \right) . \tag{14}$$

The double sum in this expression equals $nn'\theta^2L^2$; inserting this and (12) into (13) gives

$$\langle [F(\theta)-F(0)]^2\rangle = [\tfrac{1}{4}\tilde{K}^2L^4\theta^4 + K_2L^2\theta^2](k_BT)^2 . \tag{15}$$

The first term in (15) is just the square of the mean [Eq. (11)] of $\Delta F(\theta)$; thus the second term gives the fluctuations about this mean. Note that as $L\to\infty$ for *fixed* θ, these fluctuations become negligible relative to the mean; i.e., the system is "self-averaging."

Now turn to the single step case, $\theta = 1/L$. This gives the mean-squared step energy $\langle (\Delta F_S)^2\rangle = (\tfrac{1}{4}\tilde{K}^2 + K_2)(k_BT)^2$ which is finite in the high-temperature phase. This implies easy growth, or, in the superconducting problem, $V = RI$ with R finite as $I\to 0$.

Now consider the low-temperature phase. The renormalization group can be used to relate ΔF^2 in a large system to ΔF^2 in a smaller system that is easier to analyze:

$$\langle [\Delta F(\theta, L, \tilde{K}, K_2(l=0)=0, U(l=0))]^2\rangle = \langle [\Delta F(\theta L/a, a, \tilde{K}, K_2(l=\ln(L/a)), U_*)]^2\rangle . \tag{16}$$

Since the right-hand side of (16) is evaluated in a finite system (so that no infrared divergences occur), the contribution of the U term to it will be negligible as $U_*\to 0$, which happens as $T\to T_G^-$. In this limit, (16) can be evaluated using Eqs. (10) and (15), giving

$$\frac{\langle [\Delta F(\theta)]^2\rangle}{(k_BT)^2} = \tfrac{1}{4}\tilde{K}^2L^4\theta^4 + K_2\left[l=\ln\left(\frac{L}{a}\right)\right] L^2\theta^2 \cong \tfrac{1}{4}\tilde{K}^2L^4\theta^4 + \frac{C_1U_*^2}{2}\ln\left(\frac{L}{a}\right)L^2\theta^2 . \tag{17}$$

The system remains self-averaging for *fixed* θ as $L\to\infty$, as was assumed in Ref. [18].

The *single step* energy, however, diverges. Taking $\theta = 1/L$ in (17) gives

$$\frac{\langle \Delta F_S^2\rangle}{(k_BT)^2} = \frac{\langle [\Delta F(\theta=1/L)]^2\rangle}{(k_BT)^2} = \tfrac{1}{4}\tilde{K}^2 + \frac{C_1U_*^2}{2}\ln\left(\frac{L}{a}\right) , \tag{18}$$

which diverges as $L\to\infty$ like $\ln(L/a)$. Identifying Eq. (18) as $E_J^2(L)$ gives Eq. (1); Eq. (9) for $U_*(T)$ then implies Eq. (2). Given the form (1) for the energy barriers, standard nucleation arguments [23,24] give the growth velocity V_G of the crystal when that growth is driven by a crystal-fluid chemical-potential difference $\Delta\mu$. Equation (3) for $V(I)$ follows from the resultant V_G through the mapping to flux lines described earlier (in which $\Delta\mu$ is replaced by the current density J).

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[13] This experimental realization of the model was conceived by P. M. Goldbart.

[14] Strictly speaking, this model very artificially imposes perfect correlations of the bulk randomness along the surface normal (since ϕ is taken to be independent of h). However, as shown in J. Toner and D. P. DiVincenzo, Phys. Rev. B **41**, 632 (1990), Appendix A, this approximation should work well in equilibrium out to a length scale L_v exponentially long in the bulk correlation length ξ_B. As long as the plateau nucleation radius $r_c < L_v$ (as it will be down to growth rates exponentially small in ξ_B), the theory presented here remains valid.

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[21] J. L. Cardy and S. Ostlund, Phys. Rev. B **25**, 6899 (1982). My $K_2=(K-\tilde{K})/2$ in their notation.

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