Analytic Model for Scaling of Breakdown

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A theory is presented for the time-dependent breakdown of a network of spring (fuse) elements where the probability of breaking an element under load σ is σ^η. For all η, it predicts the system-size scaling of the number of broken elements at breakdown found in simulations. The breakdown is shown to be percolationlike for $\eta \leq 2$ but is due to the dominance of one large growing crack, despite the absence of a failure threshold, for $\eta > 2$. This transition in fracture behavior and in scaling at $\eta > 2$ is found to be directly related to the dependence of crack tip stress enhancement on the square root of crack size.

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The problem of fracture in random systems is receiving growing attention because of its relevance to the performance of advanced composites and its relation to random system problems in statistical mechanics [1]. The fracture in brittle systems, where the basic elements exhibit linear response up to a failure point, is particularly complicated because of its sensitivity to extreme events, i.e., the rare large crack which propagates unstably across the sample prior to any appreciable crack growth elsewhere in the material, which leads to system-size-dependent failure. Systems with time-dependent disorder are of special interest in connection with creep and fatigue and are also closely related to problems such as diffusion-limited aggregation (DLA) and dielectric breakdown [2]. To date, failure and related problems have largely been studied by simulations. But, since real systems are much larger than those accessible by simulation, it is crucial to understand the dominant mechanisms of failure and verify the nature of any size dependence observed in simulations by using analytic models.

In this Letter, we study a failure problem in which each element breaks at a rate dependent on its time-dependent stress. The local stresses vary in time because of load transfer from previously failed elements to intact elements, leading to considerable disorder in the stress fields and failure patterns. At some time, global failure occurs as the system breaks into two pieces. The system thus has an intrinsic lifetime and can only sustain a certain amount of damage prior to complete failure. Despite the apparent complexity of the global failure process, we introduce here an analytic model which captures the essence of the fracture behavior, and *predicts the observed size scaling of damage at global failure.* The model includes only the stress enhancements at the tips of any cracks and neglects all the disorder associated with longer-range stress fields and crack interactions. The *quantitative* success of our model (see Fig. 1) leads to the important and surprising conclusion that much of the disorder does *not* influence the size scaling of this failure process.

The specific time-dependent disorder problem we address here, and studied by simulation techniques [3], is as follows. A two-dimensional lattice of N_T connected springs (or fuses) is subjected to an externally applied stress. The springs are then "broken" one at a time by a probabilistic algorithm where the next spring to break is selected by (i) assigning a break probability $p_i(t)$ to each element that is proportional to the stress on that element raised to a power η, $p_i(t) = s_i(t)^\eta \Delta t$, where $s_i(t)$ is the stress on element i at time t, and $\Delta t = 1/\sum_i s_i(t)^\eta$ is the average time required to break the next element (a unit stress is assumed applied at $t = 0$), and then (ii) choosing one spring randomly but weighted by the probabilities $p_i(t)$. The time evolution is simply incremented by Δt after each break. This algorithm is identical to those used in studies of polymer failure [4] and those used to describe dielectric breakdown and DLA ($\eta = 1$) with the important exception that here broken elements may occur anywhere in the lattice and need not be connected to the single evolving tree as in dielectric breakdown [2]. In two

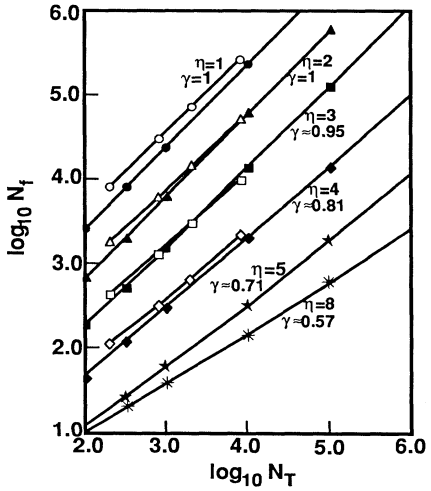

FIG. 1. Number of failed elements N_f at global failure vs system size N_T, for various η: open symbols, simulation data (Ref. [3]); solid symbols, present theory. Data for $\eta = 1,2,3,4$ are vertically offset by 2.0, 1.5, 1.0, and 0.5, respectively. Approximate scaling exponents γ for theory results are indicated.

extreme limits the behavior of the system is easily understood [3]: For $\eta \to 0^+$ ("screened" percolation) global failure occurs only after a finite fraction $N_f/N_T > 0$ of elements have failed and hence $N_f \propto N_T$; for $\eta \to \infty$ the element under the largest stress is always selected to fail, and so failure corresponds to the growth of a single crack across the sample since the highest stress always occurs at the tip of the largest (in this case, the only) crack and hence $N_f \propto N_T^{1/2}$. The scaling of N_f with N_T for other values of η has been studied using simulations by Hansen, Roux, and Hinrichsen [3] for square fuse networks loaded by busbars along the diagonals and, less systematically, by the present authors for triangular spring networks [5]. The results of Hansen, Roux, and Hinrichsen are shown in Fig. 1 for $\eta = 1$–4. For $\eta = 1$ and 2 the exponent γ of $N_f \propto N_T^{\gamma}$ is 1 within statistical errors, while for $\eta = 3$ and 4 the exponents are clearly smaller than 1 and decreasing with increasing η.

To make progress on understanding the above size scaling, we consider only the stress transfer to the z neighbors at the tips of any existing cracks, where the stress is highest. The stress at the tip of a size-c crack relative to the applied stress is denoted σ_c, and depends on c as $\sigma_c \sim c^{1/2}$, a well-known result in fracture mechanics [6]. We further neglect shielding of elements by cracks and crack interactions. Then, the cracks become independent and each element may be classified as either (i) part of an existing crack, (ii) at one of the z tip elements of an existing crack, or (iii) an "isolated" element under only the applied load. With this simplification, the main focus is on the consequences of the nonlinear dependence σ_c^{η} of the relative crack growth rates versus crack size, which will be shown to capture key aspects of the global failure. Evolution of the entire system is then easily described by a master equation for the evolution of the crack size distribution $N(c,t)$ where $N(c,t)$ is the number of cracks of size c (a contiguous line of c failed elements) at time t,

$$\frac{dN(c,t)}{dt} = \alpha_{c-1} N(c-1,t) - \alpha_c N(c,t), \quad c \geq 1, \tag{1a}$$

$$N(0,t) = N_T - \sum_{c \geq 1} (z+c) N(c,t), \tag{1b}$$

where $\alpha_c = z\sigma_c^{\eta}$. The terms on the right-hand side of (1a) are the fluxes into and out of the size-c cracks due to growth from $c-1$ to c, and from c to $c+1$. The unfailed, nontip elements are "cracks" of size $c = 0$ with $\alpha_0 = 1$.

Given Eqs. (1), global failure in a volume N_T with linear span $L = N_T^{1/2}$ is rigorously defined as the point at which the number of cracks of size $c \geq L$ reaches unity; i.e., when the so-called "risk of rupture" function $R(c,t) = \sum_{c' \geq c} N(c,t)/N_T$ attains $R(L,t_f) = 1/N_T$ at some failure time t_f. The number of failed elements at global failure is then $N_f = \sum c N(c,t_f)$. It also proves useful, however, to monitor the damage evolution through the decreasing mean-field elastic modulus $E(t)$,

$$E(t) = 1 - \beta \sum c^2 N(c,t)/N_T \tag{2}$$

(with $\beta = \frac{16}{9}$ in a continuum model [7]), and to associate global failure with the point $E(t) = 0$. The implications of $E(t) \to 0$ will be considered below. From Eqs. (1) and the failure conditions $R(L,t_f) = 1/N_T$ and/or $E(t) \to 0$, the scaling of N_f vs N_T with η is as follows.

For $\eta \leq 2$, solutions to Eqs. (1) show that global failure occurs via a proliferation of small defects $\ll L$ such that $E(t) \to 0$, and $N_f \propto N_T$. At "failure" no cracks are even close to the linear length L of the system but the number of elements failed or at crack tips is an appreciable fraction of N_T. Hence, with the inclusion of specific crack interactions all these defects would link up to form a macrocrack upon the failure of only a limited number of additional elements, much as the infinite percolation cluster forms rapidly as $p \to p_c$. $E(t) = 0$ in this noninteracting crack model thus gives the approximate point at which crack linking, followed by global failure, occurs. So, while the present theory is not quantitatively accurate in the final stages of failure for $\eta \leq 2$, it preserves the scaling relationship $N_f \propto N_T$. The scaling for $\eta \leq 2$ may be understood by considering the short-time solution to Eqs. (1),

$$N(c,t) \simeq N_T \left[\prod_{i=0}^{c-1} \alpha_i^{\eta} \right] t^c/c!, \quad \alpha_c t/c < 1. \tag{3}$$

For $\eta = 2$ and $\sigma_c = Ac^{1/2}$ (A is a constant) the condition $\alpha_c t/c < 1$ reduces to a time limit $t \leq 1/zA^2$ below which Eq. (3) is approximately valid for all c. Furthermore, the product of the α_i^2 is proportional to $(c-1)!$ so that $N(c,t) \simeq t^c/c$. It is easy to then show that the time required to obtain $E(t) = 0$ is less than $1/zA^2$ and, moreover, that the number of failed elements N_f is a fixed fraction of N_T, i.e., $\gamma = 1$, independent of the value of β entering the mean-field $E(t)$. The scaling for $\eta = 2$, and hence $\eta \leq 2$, is thus a direct consequence of the $\sigma_c \sim c^{1/2}$ dependence.

For $\eta > 2$, the short-time solution to Eqs. (1) [Eq. (3)] is only valid, at fixed time, up to some maximum crack size. For larger crack sizes the distribution $N(c,t)$ begins to scale algebraically [8] with c, and for very large cracks approaches $N(c,t) \sim \alpha_c^{-1} \sim c^{-\eta/2}$. Hence, well prior to attaining $E(t) = 0$ by a proliferation of small defects, the quantity $R(c,t)$, which can be expressed as $R(c,t) = \int_0 dt' \alpha_{c-1} N_{c-1}/N_T$, becomes a very slowly varying function of c for sizes c such that $\alpha_c t/c \gg 1$. Thus, the "largest crack" c^* at time t, as determined by $R(c^*,t) = 1/N_T$, rapidly increases from small sizes $\ll L$ to the size L in a very short time near t_f. Figure 2(a) shows $R(c,t)$ for $\eta = 4$, $N_T = 10^4$ ($A = 1 + 1/z$, $z = 4$) at various times during the damage evolution, with a graphical determination of c^*. The rapid change in c^* corresponds to the rapid propagation of a single crack across the sample with limited crack growth elsewhere, a runaway phenomenon which cannot occur for $\eta \leq 2$. The runaway breakdown essentially cuts off the damage evolution occurring at smaller crack sizes and leads to the sublinear

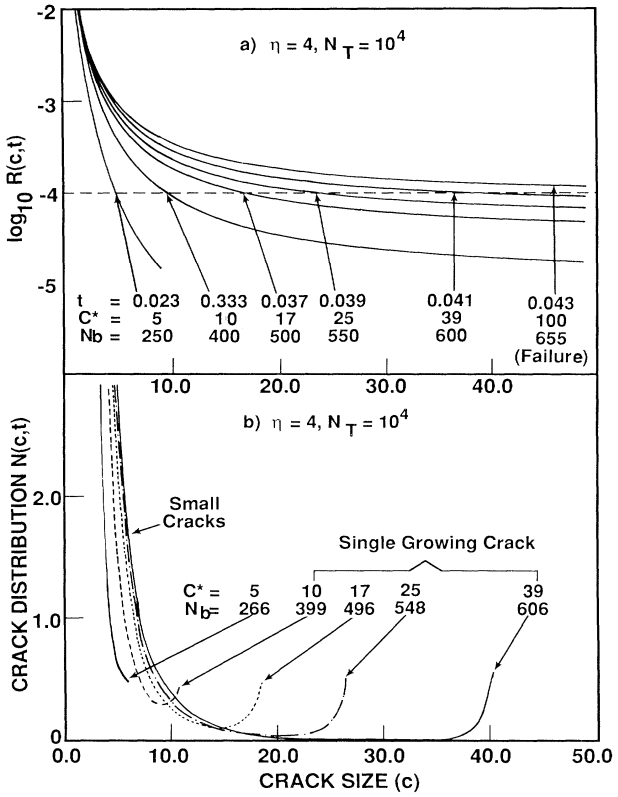

FIG. 2. (a) $\log_{10}$ of the risk of rupture $R(c,t)$ vs c for various t near failure as predicted from Eq. (1) for $\eta=4$, $N_T=10^4$. The largest crack c^* at each time occurs at $R(c^*,t)=1/N_T$, as shown, and grows rapidly as $t\rightarrow t_f$ due to the slow c dependence of $R(c,t)$. Also shown are the numbers of failed elements N_b at each time. (b) Evolution of the crack distribution $N(c,t)$ vs c for various t near failure as predicted from Eqs. (1) and (4), for $\eta=4$, $N_T=10^4$, and $p_{\min}=0.5$. Note the emergence and rapid propagation of a single crack as $t\rightarrow t_f$. The number of failed elements N_b at each time compares well with those in (a) for the same "largest crack" size c^* [i.e., the same c^* satisfying $\sum_{c\geq c^*}N(c,t)=1$].

scaling $N_f\propto N_T^{\gamma}$ ($\gamma<1$). The sublinear scaling is also seen clearly from Fig. 2(a): Increasing N_T lowers the $1/N_T$ line and reduces the time, or fraction of elements broken, required to obtain the failure condition $R(L,t_f)=1/N_T$. The value of the "scaling exponent" γ as a function of η involves the details of precisely how $N(c,t)$ approaches its asymptotic scaling of $N(c,t)\sim\alpha_c^{-1}$ for $c\simeq L$, and analytic expressions for $N(c,t)$, and ultimately γ, can be obtained under the approximation $N(0,t)\simeq N_T$.

Before presenting detailed numerical results showing the quantitative accuracy of this theory, we summarize the above general analysis. Solutions to the crack evolution Eqs. (1a) and (1b) show that (i) for $\eta\leq 2$ failure occurs almost as in the screened percolation ($\eta=0^+$) problem, with the $N_f\propto N_T$ scaling due to a crack growth rate $\alpha_c=z\sigma_c^{\eta}$ not increasing with c, while (ii) for $\eta>2$ the rate α_c increases with c, leading to an algebraic tail in $N(c,t)$ after some time, subsequent runaway breakdown, and a sublinear scaling $N_f\propto N_T^{\gamma}$, $\gamma<1$.

Note that Eqs. (1) are valid for any system size N_T and all crack sizes c, including cracks such that $N(c,t)<1$, i.e., cracks which do not exist *on average* in a sample of size N_T. To make quantitative contact with simulation results (and also to reduce the computational effort considerably), we consider Eqs. (1) valid only up to crack size $c_{\max}-1$, where $N(c_{\max}-1,t)\geq p_{\min}$ and $N(c_{\max},t)<p_{\min}$ and $p_{\min}\simeq 1$. The largest crack of size $c_{\max}$ then evolves as

$$\frac{dN(c_{\max},t)}{dt}=\alpha_{c_{\max}-1}N_{c_{\max}-1}. \tag{4}$$

That is, we do not allow the size-$c_{\max}$ crack, which occurs with probability less than $p_{\min}$ at time t in the size-N_T sample, to grow in time to size $c_{\max}+1$. $c_{\max}$ does increase with time by 1 unit as $N(c_{\max},t)$ crosses the value $p_{\min}$ by the evolution Eq. (4). By this procedure, the small but finite probability of occurrence of cracks of size $>c_{\max}$ is continually redistributed among the cracks of size $\leq c_{\max}$. The precise choice of the cutoff proability $p_{\min}\simeq 1$ has no effect on the results for $\eta\leq 2$ and no effect on the scaling for $\eta>2$. For $\eta>2$, $p_{\min}$ is physically chosen so that the large crack that emerges and propagates to cause failure occurs with unit probability.

For specific comparisons to the data of Hansen, Roux, and Hinrichsen we utilize the approximate form $\sigma_c=(1+1/z)c^{1/2}$, which captures the crucial c dependence of σ_c and $z=4$. Our results for N_f vs N_T are shown in Fig. 1 for $\eta=1$-4 with $p_{\min}=0.5$. As expected from our analysis, for $\eta\leq 2$ we find $N_f\propto N_T$ for all N_T and for $\eta>2$ we find the sublinear scaling $N_f\propto N_T^{\gamma}$ ($\gamma<1$), with $\gamma\simeq 0.95$ for $\eta=3$ and $\gamma\simeq 0.81$ for $\eta=4$ over the range of simulation sizes, but with an increasing trend in γ with increasing N_T. Results for $\eta=5$ ($p_{\min}=0.5$) and 8 ($p_{\min}=1.0$) are also shown; here the increasing trend in γ values is clearly evident, but $\gamma\simeq 0.71$ for $\eta=5$ and $\gamma\simeq 0.57$ for $\eta=8$ provide reasonable fits for $N_T\leq 10^4$. The emergence and propagation of the single crack in the crack distribution $N(c,t)$ is shown in Fig. 2(b) for the same values of η, N_T, and c^* as in Fig. 2(a). Use of Eq. (4) does not appreciably alter the emergence and propagation of the large crack, nor the number of failed elements N_b during the evolution of damage [compare the N_b in Figs. 2(a) and 2(b)]. *Overall, we find excellent agreement with the scaling relations and with the actual magnitudes of N_f obtained via simulation, with no adjustable parameters.*

In a future paper, we will discuss the richness of our simple model in more detail, and include (i) the scaling of the failure time t_f with system size, (ii) the dependence of γ on the form of σ_c for $\eta>2$, (iii) the insensitivity of the size scaling to crack linking interactions, which can be included exactly in some cases, and to mean-field

crack interactions, and (iv) the relation of our model to that of Phoenix and Tierney [9], which exhibits some similar features.

The quantitative agreement of our theory with simulation results leads to the surprising conclusion that crack interactions *do not* play a key role in the failure for $\eta > 2$, and for $\eta \leq 2$ are probably relevant only in the final stages of global failure. Instead, the dominant feature is the stress enhancement at the tip of an isolated crack, and the scaling transition for $\eta > 2$ stems directly from the dependence of that stress on the square root of crack size. The evolution of damage in quenched-disorder models [1], where the elements are assigned breaking thresholds *a priori,* may thus also be approachable with a similar neglect of specific crack interactions. For $\eta > 2$, we also find that failure occurs by the dominant propagation of a single crack, despite the absence of any threshold instability condition in the problem. In closing, we reiterate that since these size effects are not small corrections to the "thermodynamic limit" $N_T \rightarrow \infty$ but instead dominate the failure of the system, it is imperative to understand the origin of failure processes and their size scaling through tractable analytic models such as the present model.

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