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Scaling Properties of Localization in Random Band Matrices: A σ-Model Approach

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We reduce a random-band-matrix (RBM) problem to a one-dimensional, nonlinear, supersymmetric, σ model. This reduction becomes exact in the limit $b \rightarrow \infty$, b being the effective bandwidth. We prove that b^2/N, N being the matrix size, is the relevant scaling parameter. When the mean value of diagonal elements increases linearly along the diagonal an extra scaling parameter arises. These conclusions are in agreement with recent numerical results.

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Among other ensembles of random matrices that of band matrices is under intensive investigation at present. Random band matrices (RBM) were claimed to be relevant for the explanation of properties of quantum systems whose classical counterparts display chaotic behavior [1]. The kicked rotator should be mentioned as a generic example [2]. Besides, RBM arise in the course of investigation of the conductance fluctuations of thin disordered slabs by a transfer-matrix method [3].

Physical applications of RBM dictate the concentration of interest on localization properties of their eigenvectors. When the bandwidth b is sufficiently small all eigenvectors are localized and eigenvalues turn out to be noncorrelated. This is the intrinsic property of spectra of quantum systems integrable in the classical limit [4]. In the opposite case $b \propto N$, it is quite clear that the RBM ensemble does not differ practically from the Gaussian one characterized by delocalized eigenvectors and correlated eigenvalues (modeling spectral properties of "chaotic" quantum systems [4]). Therefore, the RBM ensemble in a whole range of bandwidths $1 \leq b \leq N$ seems to be suitable for interpolating between "integrable" and chaotic regimes of time-reversal invariant quantum systems [5].

The present understanding of statistical properties of RBM spectra is based mainly on results of numerical simulations. As it was convincingly demonstrated in Ref. [6], there is the scaling parameter $\tilde{x} = b^2/N$ governing the behavior of the system with zero mean value of random matrix elements. Qualitative arguments about the origin of such a parameter were presented in [6,7]. In the recent paper [7], another class of RBM (that with the mean value of diagonal elements linearly increasing along the diagonal) was considered. It turned out that an extra scaling parameter enters the problem.

In contrast to numerous computer investigations of RBM, analytical results are rather scarce. Besides the exactly soluble case of tridiagonal matrices [8], the only proven property is the validity of the Wigner semicircle law for the density of states (DOS) when the bandwidth b increases with the matrix size N as $b \propto N^{\beta}, \beta > 0$ [9].

In the present paper we perform the analytical investigation of the RBM properties within the framework of the supersymmetric approach. This method proved to be quite powerful as applied to Gaussian ensembles [10,11] and that of sparse random matrices [12]. By using the procedure analogous to that introduced by Schäfer and Wegner [13], we reduce the RBM problem in the limit $N \gg 1$, $b \gg 1$ to the one-dimensional supersymmetric σ model. This very model was considered by Efetov and Larkin (see review [10]), as applied to the investigation of electron localization in wires. Exploiting their results we prove the scaling behavior found in [6,7].

Our method proves to be useful for understanding localization in more complicated RBM as well. Using an "electron transport" formulation of the problem, suggested by our approach, we succeeded both in explaining the nature of a scaling parameter that arises in RBM with linearly increasing diagonal elements [7] and in predicting the correct asymptotic form of a scaling function

[7,14].

Let us consider an ensemble of random bandlike $N\times N$ ($N\gg 1$) matrices $\hat{H}$. The matrix elements $H_{ij}=H_{ji}$ are assumed to be real independent random numbers distributed according to the Gaussian law with zero mean value and variances

$$\overline{H_{ij}^2}=\tfrac{1}{2}A_{ij}[1+\delta_{ij}]\,, \qquad (1)$$

where $A_{ij}\equiv a(|i-j|)$ depends only on the distance $r=|i-j|$. The function $a(r)$ is assumed to decrease sufficiently fast at $r>b$, b being the effective bandwidth.

To investigate localization properties of such an ensemble, let us consider the following correlation function:

$$\mathcal{K}_{lm}=\langle l|(E+i\epsilon-\hat{H})^{-1}|l\rangle\langle m|(E-i\epsilon-\hat{H})^{-1}|m\rangle\,. \qquad (2)$$

The exponential decrease of the connected part of this correlation function at large distances $r=|m-l|$ determines the localization length.

Introducing for every "site" j (corresponding to a matrix row) a supervector $\phi_j^\dagger=(S_j^1,S_j^2,\chi_j^\dagger,-\chi_j)$ with two commuting components S_j^1,S_j^2 and two Grassmanian ones χ_j^*,χ_j, one can express advanced and retarded Green's functions

$$G_s(E,\epsilon,j)\equiv\langle j|[E+i(-1)^{s-1}\epsilon-\hat{H}]^{-1}|j\rangle$$

in terms of "superintegrals" [10]:

$$G_s(E,\epsilon,j)=(-1)^s\frac{i}{4}\int\prod_{k=1}^{N}[d\phi_k](\phi_j^\dagger\hat{K}\phi_j)\exp\left[-(-1)^s\frac{i}{2}\sum_{i,j}\phi_i^\dagger D_{ij}\phi_j\right], \qquad (3)$$

where

$$[d\phi_j]=dS_j^1\,dS_j^2\,d\chi_j^*\,d\chi_j,\quad D_{ij}=[E-i\epsilon(-1)^s]\delta_{ij}-H_{ij},\quad K=\mathrm{diag}(1,1,-1,-1)\,.$$

Then the correlation function $\mathcal{K}_{lm}$ is given before the averaging by the following expression:

$$\mathcal{K}_{lm}=\frac{1}{4}\left.\frac{\partial^2 Z(J)}{\partial J_l^1\,\partial J_m^2}\right|_{J=0}, \qquad (4)$$

$$Z(J)=\int\prod_{i,s}[d\phi_i^{(s)}]\exp\left[\frac{i}{2}\sum_{ij,s}\phi_i^{(s)\dagger}\hat{D}_{ij}^sL_s\phi_j^{(s)}\right],\quad i=1,\ldots,N,\quad s=1,2\,,$$
$$D_{ij}^s=\{[E\hat{I}+i\epsilon L_s\hat{I}+J_i^s\hat{K}]\delta_{ij}-H_{ij}\},\quad L_s=(-1)^{s-1}\,. \qquad (5)$$

Averaging Eq. (5) over the disorder we have

$$Z_{\rm av}\equiv\mathcal{N}\int\prod_{i,j}dH_{ij}\exp[-\tfrac{1}{2}H_{ij}^2(A_{ij})^{-1}]Z(J)$$
$$=\int\prod_j[d\phi_j]\exp\left[-\frac{1}{8}\sum_{i,j}(\phi_i^\dagger\hat{L}\phi_j)^2A_{ij}+\frac{i}{2}\sum_i\phi_i^\dagger[E\hat{L}+i\epsilon+\hat{J}_i\hat{K}\hat{L}]\phi_i\right], \qquad (6)$$

where we united two supervectors $\phi_i^{(1)\dagger},\phi_i^{(2)\dagger}$ into a single eight-component supervector $\phi_i^\dagger=(\phi_i^{(1)\dagger},\phi_i^{(2)\dagger})$ with $\hat{L},\hat{K},\hat{J}$ becoming 8×8 supermatrices [10–12].

To proceed further we decouple variables ϕ_i connected with different sites i by means of the Hubbard-Stratonovich transformation. We obtain

$$Z_{\rm av}=\int\prod_j[d\phi_j]\exp\left[\frac{i}{2}\sum_i\phi_i^\dagger(E\hat{L}+i\epsilon+\hat{J}_i\hat{K}\hat{L})\phi_i\right]$$
$$\times\int\prod_i d\sigma_i\exp\left[-\frac{1}{2}\mathrm{Str}\sum_{i,j}\hat{\sigma}_i(A^{-1})_{ij}\hat{\sigma}_j-\frac{i}{2}\sum_i\phi_i^\dagger\hat{L}^{1/2}\hat{\sigma}_i\hat{L}^{1/2}\phi_i\right], \qquad (7)$$

where $\hat{\sigma}_i$ are 8×8 supermatrices and the symbol Str stands for the supertrace. Changing the order of integration in Eq. (7) and performing the integration over ϕ_i we come to

$$Z_{\rm av}=\int\prod_i d\hat{\sigma}_i\exp\{-S[\hat{\sigma},\hat{J}]\}\,,$$
$$S[\hat{\sigma},\hat{J}]=\tfrac{1}{2}\,\mathrm{Str}\sum_{i,j}\hat{\sigma}_i(A^{-1})_{ij}\hat{\sigma}_j+\tfrac{1}{2}\,\mathrm{Str}\sum_i\ln(E-\hat{\sigma}_i+i\epsilon\hat{L}+\hat{J}_i\hat{K})\,. \qquad (8)$$

Making a shift, $\hat{\sigma}_i\to\hat{\sigma}_i+i\epsilon\hat{L}+\hat{J}_i\hat{K}$, and differentiating with respect to the source matrix $\hat{J}$, we get from Eq. (4) the following expression for the averaged correlator $\overline{\mathcal{K}}_{lm}$ at $\epsilon \ll 1$:

$$\mathcal{K}_{lm}=\frac{1}{4}\int\prod_i d\hat{\sigma}_i\left[\sum_j (A^{-1})_{lj}\,\mathrm{Str}\hat{K}\hat{\sigma}_j^{11}\right]\left[\sum_k (A^{-1})_{mk}\,\mathrm{Str}\hat{K}\hat{\sigma}_k^{22}\right]e^{-S_0[\hat{\sigma}]},$$
$$S_0[\hat{\sigma}]=\frac{1}{2}\,\mathrm{Str}\left[\sum_{i,j}\hat{\sigma}_i(A^{-1})_{ij}\hat{\sigma}_j+\sum_i \ln(E-\hat{\sigma}_i)+\frac{2i\epsilon}{B_0}\sum_i\hat{\sigma}_i\hat{L}\right], \tag{9}$$

where $B_0=\sum_i A_{ij}$ and we used the block notation

$$\hat{\sigma}=\begin{bmatrix}\hat{\sigma}^{11}\hat{\sigma}^{12}\\ \hat{\sigma}^{21}\hat{\sigma}^{22}\end{bmatrix}$$

stemming out of the following decomposition of the supervector, $\phi^\dagger=(\phi^{(1)\dagger},\phi^{(2)\dagger})$, each $\hat{\sigma}^{ss'}$ being a 4×4 supermatrix conjugated to $\phi^{(s)}\phi^{(s')\dagger}$.

To make further consideration as clear as possible, let us specify $A_{ij}\equiv a(|i-j|)$ to be of the following form:

$$a(r)=a(-r)=(2/b)\exp(-|r|/b). \tag{10}$$

The main advantage of such a choice is that the matrix A^{-1} entering Eq. (8) is tridiagonal (when solving a one-dimensional Ising model this fact was used in [15]):

$$(A^{-1})_{ij}=\frac{b}{2}\frac{1}{1-e^{-2/b}}[(1+e^{-2/b})\delta_{ij}-e^{-1/b}(\delta_{i,j+1}+\delta_{i,j-1})]. \tag{11}$$

As it is shown below, our main conclusions are insensitive to the specific form of the matrix A.

Substituting Eq. (10) into Eq. (8) and considering $b\gg 1$ we get

$$S[\hat{\sigma}]|_{J=0}=\sum_i \mathrm{Str}[\tfrac{1}{8}b^2(\hat{\sigma}_i-\hat{\sigma}_{i+1})^2+\tfrac{1}{8}\hat{\sigma}_i^2+\tfrac{1}{2}\ln(E-\hat{\sigma}_i)+\tfrac{1}{4}i\epsilon\hat{\sigma}_i\hat{L}]. \tag{12}$$

To calculate correlation functions we use the saddle-point approximation justified by two large parameters $N\gg 1, b\gg 1$.

At this point we should note that a closely analogous procedure can be used for the calculation of the averaged one-site Green's function

$$\langle i|(E+i\epsilon-\hat{H})|i\rangle,$$

the imaginary part of which gives the density of states. In this case we can restrict ourselves by introducing only one four-component supervector per site. The resulting expression for the action $S[\hat{\sigma}]$ would coincide with Eq. (12) with the only replacement of the matrix $\hat{L}$ by the identity matrix $\hat{I}$. Then the saddle-point equation has the single solution

$$\hat{\sigma}=\tfrac{1}{2}[E-i(8-E^2)^{1/2}]\hat{I}, \tag{13}$$

independent of the site index i, and for the density of states we get

$$\rho=(1/2\pi)(2-\tfrac{1}{4}E^2)^{1/2},\quad E^2\le 8, \tag{14}$$

and $\rho=0$ otherwise, which is nothing but the famous Wigner semicircle law obtained for the case of large band matrices in [9].

In contrast, when calculating the density-density correlator, we have to choose the solution of the saddle-point equation in the form

$$\hat{\sigma}_i=\hat{T}^{-1}\hat{\sigma}_0\hat{T},\quad \hat{\sigma}_0=\tfrac{1}{2}[\hat{I}E-i\hat{L}(8-E^2)^{1/2}], \tag{15}$$

dictated by the convergence requirement (see Refs. [10,11,16] for a detailed discussion). Here matrices $\hat{T}$ satisfy the condition $\hat{T}^+\hat{L}\hat{T}=\hat{L}$, determining the graded Lie group UOSP(2,2/2,2). Thus, in the limit $N\to\infty$, $b\to\infty$ only the manifold (15) contributes to the integral (9).

Turning to the continuous limit and neglecting at $r=|l-m|\gg 1$ a difference between $\hat{\sigma}_l$ and $\hat{\sigma}_{l\pm 1}$, we get the following expression for the correlation function at the distance x:

$$\mathcal{K}(x)=\left[\frac{\pi\rho}{4}\right]^2\int D\hat{Q}(x)\mathrm{Str}[\hat{K}\hat{Q}^{11}(0)]\times\mathrm{Str}[\hat{K}\hat{Q}^{22}(x)]e^{-S[\hat{Q}]}, \tag{16}$$

where

$$S[\hat{Q}]=\tfrac{1}{2}\int dx\{b^2d^3(\pi\tilde{\rho})^2\mathrm{Str}(\partial_x\hat{Q})^2+\epsilon\pi\tilde{\rho}\,\mathrm{Str}[\hat{Q}(x)\hat{L}]\} \tag{17}$$

and the integration in Eq. (16) goes over matrices $\hat{Q}(x)=\hat{T}^{-1}(x)\hat{L}\hat{T}(x)$, forming the graded coset space UOSP(2,2/2,2)/UOSP(2/2)×UOSP(2/2) [11]. Here we wrote explicitly an irrelevant parameter d measuring the distance between neighboring sites and introduced the density per unit length $\tilde{\rho}=\rho/d$.

The action (17) defines the one-dimensional supersymmetric σ model investigated in a context of electron localization in wires [10]. The parameters entering expression (17) are related to the bare diffusion constant D_0 and the frequency ω as follows:

$$D_0 \sim 4\pi\tilde{\rho} b^2 d^3, \quad \omega = 2i\epsilon . \tag{18}$$

In Ref. [10] the complete localization of states for this model was proved and the following expression for the localization length was found: $\xi_{\rm loc} = 4\pi\tilde{\rho} D_0$. Substituting relations (18) into this formula we obtain the final expression for the dimensionless localization length (measured in units of intersite distance d):

$$l_{\rm loc} \equiv \xi/d = (4\pi\rho b)^2 = (8 - E^2) b^2 . \tag{19}$$

Applying to the present case a one-parameter scaling hypothesis put forward for disordered conductors in [17], we conclude that all correlation properties of RBM spectra should be determined by the single scaling parameter x equal to the ratio of the localization length to the matrix size: $\tilde{x} = l_{\rm loc}/N \propto b^2/N$. This explains the scaling behavior observed in numerical simulations [6].

This conclusion is highly insensitive to the specific form of the function $a(|i-j|)$, given its exponential decrease at the distances $r \equiv |i-j| > b$ [we assume the normalization condition $B_0 = \sum_{r=-\infty}^{\infty} a(r) \sim 1$ at $b \to \infty$]. In general, we should transform the first term in the action, Eq. (8), going to the momentum representation and restricting ourselves to the lowest-order terms in a small-momentum expansion. The resulting expression for the action has the same form

$$S[\hat{Q}] = \frac{\pi\tilde{\rho}}{8} \int dx \{ D_0 \mathrm{Str}(\partial_x \hat{Q})^2 - 2i\omega \mathrm{Str}[\hat{Q}(x)\hat{L}] \} , \tag{20}$$

where now

$$\rho \equiv \tilde{\rho} d = (1/\pi B_0)(2B_0 - E^2)^{1/2} ,$$
$$D_0 = \pi\rho B_2 ,$$
$$B_2 = \tfrac{1}{2} \sum_{r=-\infty}^{\infty} a(r) r^2 \propto b^2 . \tag{21}$$

That leads to the following general expression for the localization length:

$$l_{\rm loc} = 4\pi\tilde{\rho} D_0/d = (4/B_0^2)(2B_0 - E^2) B_2 \propto b^2 . \tag{22}$$

So far we considered RBM with zero mean value of matrix elements. Another class of RBM (that with linearly increasing mean value of diagonal elements: $\overline{H}_{ij} = \beta i \delta_{ij}$) was considered in recent work [7]. In our approach that results in the substitution $E - \beta i$ for E in Eqs. (5)-(8). Such a substitution is equivalent to adding the uniform electric field $F = \beta/d$ at the corresponding electron-transport problem. It is obvious that this procedure introduces a new length scale $\xi_{\rm el} = \Delta/F$, where $\Delta = 2(2B_0)^{1/2}$ is the width of the electron energy band in the absence of an electric field. This length arises when we consider a cross section of the energy band locally tilted by the electric field: $-\Delta/2 + \beta i \le E \le \Delta/2 + \beta i$ for an energy level $E = $const. Therefore, a new scaling parameter $y = \beta b^2 \propto \xi_{\rm loc}/\xi_{\rm el}$ enters the problem in addition to the former parameter $\tilde{x}$.

If we define an "effective localization length" $L^{\rm eff}$ as that containing the most of an eigenvector normalization (e.g., the "entropic localization length," Refs. [6,7]), it should be of the order of $\xi_{\rm loc}$ at $y \ll 1$ and of the order of $\xi_{\rm el}$ in the opposite limiting case. Therefore, $L^{\rm eff}$ has the following scaling form: $L^{\rm eff} = b^2 f(y)$; $f(y) \sim 1$ if $y \ll 1$ and $f(y) \sim \Delta/y$ if $y \gg 1$.

We should note that in Ref. [7] a different normalization of matrix elements was used: $a(r) \sim 1$ instead of our choice $a(r) \sim b^{-1}$. As a result, the parameter $\Delta \propto B_0^{1/2} \propto b^{1/2}$ and the ratio $\xi_{\rm loc}/\xi_{\rm el}$ becomes proportional to the combination $\beta b^{3/2}$ coinciding with the scaling parameter introduced in Ref. [7]. Meanwhile, the fit $f(y) \sim y^{-2/3}$ at $y \gg 1$ used in [7] obviously contradicts the correct expression $f(y) \sim y^{-1}$ [14].

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