# Remarks on Unitarization Methods

Tran N. Truong[(a)]
*Centre de Physique Theorique, Ecole Polytechnique, 91128 Palaiseau, France*
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It is shown that the unitarized $P$-wave Nambu-Goldstone scattering amplitude, constructed from the one-loop chiral perturbation theory by the $K$-matrix method, does not have a resonant behavior, but has instead one nearby pair of complex-conjugate poles on the physical sheet and should be rejected. This result is in contrast with other unitarization schemes such as the Padé, the inverse amplitude, and the $N/D$ methods, which produce a $P$-wave resonance and have the correct analytic property.

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It is the consensus of many experts working on the strongly interacting Higgs sector that the physical consequences of the unitarized scattering amplitudes of the Nambu-Goldstone (NG) bosons constructed from one-loop chiral perturbation theory (CPTh) are not unique and therefore unreliable. This feeling is reflected in many recent calculations on the scattering of the longitudinal component of the $W$ and $Z$ bosons for a strongly interacting Higgs boson, where the question of unitarity is ignored. A frequently cited example is the case of the unitarized $P$-wave $W_LW_L$ scattering amplitude that is constructed from one-loop CPTh. On the one hand, if one used the Padé method, with a correct sign for the $O(p^4)$ term as given by, for example, the derivative expansion of the effective action of the underlying theory, a $P$-wave resonance in $W_LW_L$ scattering could be generated; on the other hand, if one used the $K$ matrix, whatever was the sign of the $O(p^4)$ term, a $P$-wave resonance could not be generated. Because the existence of a low-energy $P$-wave resonance, which should be a clear signal at the CERN Large Hadron Collider (LHC) and the Superconducting Super Collider (SSC), could help to distinguish an elementary from a composite heavy Higgs boson, and because huge expenses and efforts are involved in the construction of the LHC and SSC to study the Higgs sector of the standard model, the controversy on the validity of the unitarization scheme should not be settled by consensus, but only by the basic principles of physics.

The purpose of this Letter is to give a critical examination of the known unitarization schemes. The basic principles are the analyticity (or causality) and unitarity. The following unitarization methods are studied: the Padé, inverse amplitude, $N/D$, and $K$-matrix methods. The main conclusion is that unlike other methods, the $K$ matrix, as it is usually used, could violate the Hilbert transform (dispersion relation) and should be modified or discarded. The $K$-matrix unitarized $P$-wave amplitude has a pair of complex-conjugate poles on the physical sheet with large residues, near the physical region, and therefore should be rejected.

Let us review the $P$-wave $f_1(s)$ one-loop chiral perturbation calculation which was first given by Lehmann [1] in 1972 and later by others [2]:

$$f_1(s)=(s/96\pi v^2)(1+(s/v^2)\{\tfrac{3}{4}E_R(\nu)-G_R(\nu)+(4\pi)^{-2}[-\tfrac{1}{18}+i\pi\theta(s)/6]\}), \quad (1)$$

where $v$ is equal to 93 MeV in QCD, and 246 GeV in $WW$ scattering. The dimensional-regularization-scheme version is presented here; $E_R(\nu)$ and $G_R(\nu)$ are the renormalized $O(p^4)$ parameters which depend on the scale $\nu$. The combination $\frac{3}{4}E_R(\nu)-G_R(\nu)$ is, however, independent of the scale $\nu$ and can be considered as a parameter in the calculation. (It is equal to $N_c/24\pi^2$ in QCD quark loop calculations, where $N_c$ is the color number.) This expression is valid in the chiral limit of the massless NG bosons. From the basic principle of field theory and causality [3], one can show that the partial-wave amplitude is a real analytic function in the cut $s$ plane, with the right-hand cut (or the unitarity cut) on the real $s$ axis from $4m_\pi^2$ to $\infty$, and the left-hand cut on the real axis from 0 to $-\infty$, where $m_\pi$ is the NG boson mass, and that there are no complex singularities such as poles or cuts, or poles on the negative real $s$ axis. In the chiral limit $m_\pi=0$, the two branch points at $4m_\pi^2$ and at 0 coincide with each other which makes the study of the analytical property of the partial-wave amplitude difficult. In order to study its analytical property, it is necessary to give a small mass to the NG bosons to separate the two cuts. It will then be possible to continue the partial wave throughout the complex $s$ plane to study its analytic property, as will be shown below.

$f_1(s)$ as given by Eq. (1) satisfies only perturbative unitarity instead of the full elastic unitarity condition, $\mathrm{Im}f_1(s)=|f_1(s)|^2$, and is therefore inadequate for strong interaction physics. The perturbation series for $f_1(s)$ has to be resummed in order to satisfy exactly the elastic unitarity condition. It would be ideal if the reconstructed amplitude were free from the complex singularities or poles on the real negative axis. This is usually not possible. A well-known example is the Landau ghost which appears in the photon propagator when the geometric series for the self-energy operator is summed.

Because the unitarized partial-wave amplitudes discussed below are similar to the geometric series of the self-energy operators, unwanted singularities, which de-

stroy causality, are expected to appear in the resummed series. Because the coupling constants become large in the soft hadronic physics and also in the strongly interacting Higgs sector, these unwanted poles move dangerously close to the physical region. Since the existence of unwanted poles in a unitarized amplitude can always be removed by multiplying it by an appropriate polynomial, at the expense of violating the unitarity relation, the main issue here is the acceptable degree of violation of unitarity at the energy scale under consideration. More precisely, if an $l$th partial-wave amplitude $f_l(s)$ had an unwanted pole at $-s_g$, the pole-free amplitude $f_l^m(s)$ can be constructed, $f_l^m(s)=f_l(s)(s_g+s)/s_g$. At values of $s \ll s_g$ the modified amplitude is approximately the same as the $f_l(s)$. The question is therefore with what accuracy one is able to calculate strong interaction physics. It is an illusion to suppose that soft hadronic physics can be calculated with an accuracy better than 10% or 20%. Therefore one should only demand that $s_g$ be 5 or 10 times larger than the energy scale $s$ where the calculation is made. (An exception to this rule is the harmless nearby poles which have very small residues and hence can be subtracted out without greatly affecting the unitarity relation and the low-energy theorem.) For a given set of unitarized amplitudes constructed from the one-loop perturbative calculation, one should choose only those which have unwanted singularities that are far away.

To study the analytic property of the $P$-wave amplitude, it is necessary to separate the right- and the left-hand cuts; these are not apparent in Eq. (1). This is so because the logarithm term from the $s$ channel (the right-hand cut) cancels with the logarithm term coming from the partial-wave projection of the $t$ and $u$ channels (the left-hand cut). To be more precise, Eq. (1) can be rewritten as

$$f_1(s)=\frac{s}{96\pi v^2}+\frac{s}{96\pi v^2}\left[\frac{s}{m_v^2}-\frac{s}{96\pi^2 v^2}\left(\ln\frac{-s+\mu^2}{\alpha^2}-\ln\frac{s}{\alpha^2}\right)\right], \tag{2}$$

where $\mu^2 \to 0$, $\alpha$ is some scale factor, and $m_v^2$ is defined as

$$m_v^2=\frac{v^2}{\frac{3}{4}E_R(v)-G_R(v)-1/18(16\pi^2)}.$$

In the following, $m_v^2$ is assumed to be positive. The logarithm function is defined for $s$ approaching the real axis from above as follows: For $s>0$, the phase of $\ln(-s)$ is $-i\pi$, while that of $\ln(s)$ is zero; for $s<0$ the phase of $\ln(s)$ is $+i\pi$ while that of $\ln(-s)$ is zero. Between the gap $0 \le s \le \mu^2$ the logarithm functions are real. It is simple to see that, with this definition of the logarithm, the $P$-wave amplitude satisfies the reflection principle: $f_1^*(s)=f_1(s^*)$. The study of the singularity of $f_1(s)$ in the complex $s$ plane is now simplified to the study of its analytic property in the upper-half $s$ plane. Consider first the two simplest unitarization schemes for Eq. (2), the diagonal [1,1] Padé approximant method and the $K$-matrix approach.

*(i) Padé approximant method.*—Denote the first term on the right-hand side of Eq. (2) by $f^{(0)}(s)$, the tree amplitude, and the remaining term by $f^{(1)}(s)$, the one-loop $P$-wave amplitude, and where for simplicity the subscript 1 is dropped. In the Padé method [4,5], the unitarized partial wave can be written as

$$f^P(s)=(f^{(0)})^2/[f^{(0)}-f^{(1)}], \tag{3}$$

where the $s$ dependence is dropped for simplicity. Using Eq. (2) in Eq. (3), it follows that

$$f^P(s)=\frac{s}{96\pi v^2}\frac{m_v^2}{m_v^2-s-im_v^2 s/96\pi v^2}. \tag{4}$$

Keeping in mind that this expression has both the right- and left-hand cuts due to the logarithm terms as explained above, it can be analytically continued to the complex value of $s$ in the upper-half plane and also to the negative $s$ axis. Putting $s=x+iy$, with $y \ge 0$, the zero of the denominator of $f^P(s)$, corresponding to the pole of $f^P(s)$, must satisfy the following conditions,

$$y=-\frac{m_v^2 x}{96\pi v^2}, \quad x=\frac{m_v^2}{1+m_v^4/(96\pi v^2)^2}, \tag{5}$$

which are impossible to satisfy because $y \ge 0$. It follows that the Padé amplitude does not have complex poles in the complex $s$ plane and is therefore acceptable as the correct solution. The $P$-wave phase shift $\delta_{11}$ is given as

$$\tan\delta_{11}^P=\frac{s}{96\pi v^2}\frac{m_v^2}{m_v^2-s}.$$

It is seen that the effective range approximation given in Ref. [1] is equivalent to the Padé [1,1] diagonal approximant method. The $P$-wave resonance width satisfies the Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation [6].

*(ii) K-matrix method.*—The $K$-matrix approach consists in writing [7]

$$f^K(s)=\frac{f^{(0)}+\mathrm{Re}f^{(1)}(s)}{1-i[f^{(0)}+\mathrm{Re}f^{(1)}(s)]}, \tag{6}$$

and hence satisfies exactly the elastic unitarity relation, $\mathrm{Im}f^K(s)=|f^K(s)|^2$. If follows that

$$f^K(s)=\frac{s}{96\pi v^2}\frac{m_v^2+s}{m_v^2-i(m_v^2 s/96\pi v^2)(1+s/m_v^2)}, \tag{7}$$

where the logarithm terms are not written out explicitly. This expression is also valid for $s$ complex and also on the real axis. The $P$-wave phase shift can be computed from

this equation:

$$\tan\delta_{11}^{K}=\frac{s}{96\pi v^2}\left(1+\frac{s}{m_v^2}\right). \tag{8}$$

It is seen that the $P$-wave phase shift cannot go through 90° to produce a resonance [8]. This is the contradiction between the $K$-matrix and Padé methods. In the following, it will be shown that the $K$-matrix solution has a pair of complex-conjugate poles.

When $s=x+iy$, with $y\geq 0$, the following simultaneous equations can be solved for the positions of the poles of $f^K(s)$:

$$\begin{aligned} &y(1+4y^2/m_v^4)^{1/2}=96\pi v^2,\\ &x=\tfrac{1}{2}m_v^2[-1\pm(1+4y^2/m_v^4)^{1/2}], \end{aligned} \tag{9}$$

with $x\leq -m_v^2/2$ because $y\geq 0$. For a given value of $m_v$, the pole positions can be found by numerical methods. To have a feel, one can assume $m_v^2=8\pi^2v^2$, corresponding to the QCD theory with a $\rho$ mass of 0.82 GeV; Eq. (8) shows that $f^K$ has one pole in the upper-half $s$ plane at $s=(-1.92+i1.34)m_v^2$, and hence it also has another complex-conjugate pole in the lower-half $s$ plane. This result is in contrast with the Padé solution, Eq. (4), which has no complex singularity. Because these two poles are very near to the physical region $s=m_v^2$ and their residues are large, the unitarized $P$-wave amplitude given by the $K$ matrix must be rejected.

The $P$-wave CPTh amplitude is rather special due to the cancellation of the logarithm contributions of the right- and left-hand cuts to give $\mathrm{Re}f^{(1)}(s)$ as a polynomial. Other partial waves in CPTh and in other theories do not have this property. Then, an argument, following the same line as the inverse amplitude method to be discussed below, can be made to show that there is a conflict between the usual $K$-matrix method and the dispersion relation. This is so because if $f^K(s)$ were an analytic function as required by the general property of the partial-wave amplitude, its inverse would also be an analytic function and one should be able to write a dispersion relation for it. Now since $[f^K(s)]^{-1}=(f^{(0)}+\mathrm{Re}f^{(1)})^{-1}-i$, it is clear that its real part cannot, in general, be related to its imaginary part by the Hilbert transform or dispersion relation. As an example, one can see that $\mathrm{Re}[f^K(s)]^{-1}$ calculated from the dispersion relation, given that the imaginary part is $-1$, must have a term proportional to $\ln(s)$, apart from a polynomial due to the subtraction; the perturbative result gives, however, $\mathrm{Re}f^{(1)}$ proportional to $\ln(s)$ and hence $\mathrm{Re}[f^K(s)]^{-1}$ must have a term proportional to $1/\ln(s)$. These different behaviors of $\mathrm{Re}[f^K(s)]^{-1}$ show the inconsistency between the dispersion relation and the $K$-matrix method when $\mathrm{Re}f^{(1)}$ is not a polynomial.

The above discussion does not imply that it is impossible to modify the $K$-matrix method in order to restore the correct analyticity. In a previous publication [9], it was shown how to modify this method, using the dispersion relation, to get a solution of the linear $\sigma$ model in a simple manner; the solution presented is more general than that given by the $1/N$ expansion of the $O(2N)$ model. In fact, the modified $K$-matrix solution given in Ref. [9] is equivalent to the result obtained by the $N/D$ method to be discussed below.

Because the Padé method is constructed from the full Feynman amplitudes, i.e., with both real and imaginary parts, there is usually no difficulty with analyticity as for the $K$-matrix method. This does not mean that the Padé method is sacred, only that it is the simplest one. It is also capable of giving artifacts. One must always check that its solution is free from nearby unwanted singularities. It is now useful to examine whether other methods of unitarization, the inverse amplitude and the $N/D$ methods, will lead to the same conclusion as the Padé method.

*(iii) Inverse amplitude method.*—This method is directly related to the diagonal [1,1] Padé approximant; however, it is more general. In fact, it will be shown below that the Padé approximant is the first iteration of the nonlinear singular integral equation obtained by writing down the dispersion relation for the inverse of the partial-wave amplitude, $f_l^{-1}(s)$. Because the inverse of an analytic function is also an analytic function, apart from the additional pole contributions due to the zeros of the amplitude, one can write a dispersion relation for $f_l^{-1}(s)$. Define for this purpose $f_1(s)=(s/96\pi f^2)g^{-1}(s)$; a dispersion relation can be written for $g(s)$, assuming that it has no zeros:

$$g(s)=1+\beta s+\frac{s^2}{\pi}\int_0^\infty ds'\frac{\mathrm{Im}g(s')}{s'^2(s'-s-i\varepsilon)}+\frac{s^2}{\pi}\int_{-\infty}^0 ds'\frac{\mathrm{Im}g(s')}{s'^2(s'-s-i\varepsilon)}, \tag{10}$$

where $\beta$ is a subtraction constant and is related to $E_R$ and $G_R$. On the right-hand cut, the unitarity relation yields $\mathrm{Im}g(s)=-(s/96\pi f^2)$, while on the left-hand cut $s\leq 0$, $\mathrm{Im}g(s)=-(s/96\pi f^2)\,\mathrm{Im}f_1(s)/|f_1(s)|^2$. On the left-hand cut, $\mathrm{Im}f_1(s)$ is usually given by the analytic continuation of a sum of the partial waves from the physical region. In one-loop CPTh, $\mathrm{Im}f_1(s)=(s/96\pi v^2)^2$ on the left-hand cut. Using this approximation in Eq. (9), a nonlinear singular integral equation is obtained. As a first iteration of this integral equation, one can set perturbatively $f_1(s)=s/96\pi f^2$ for $s\leq 0$ to get

$$g(s)=1+\beta s-\frac{s^2}{96\pi^2v^2}\int_0^\infty ds'\frac{1}{s'(s'-s-i\varepsilon)}-\frac{s^2}{96\pi^2v^2}\int_{-\infty}^0 ds'\frac{1}{s'(s'-s-i\varepsilon)}, \tag{11}$$

which is simply the integral representation of the denominator of the Padé $P$-wave amplitude, Eq. (4). Although the equivalence between the Padé [1,1] approximant method and the first iteration of the integral equation for the inverse

amplitude is explicitly demonstrated for the special form of the $P$-wave amplitude, it can be shown that the proof is quite general.

*(iv) $N/D$ method.*—Instead of writing an analytic function with two cuts as the sum of two analytic functions, one with the right-hand cut and the other with the left-hand cut, as it was done for the inverse amplitude method, one can write it as the product of the two cuts: This is the $N/D$ method [3,10]. The $D$ function has only the discontinuity across the right-hand cut and the $N$ function has only the discontinuity across the left-hand cut. It is clear with the $N/D$ method that one cannot rederive the Padé amplitude because the logarithm functions due to the contribution of the two cuts, written in the $N/D$ product, cannot cancel each other. Some approximation can be made in order to show that, with the correct sign of the $O(p^4)$ term as given above, a $P$-wave resonance can be generated. Writing $f_1(s)=N(s)/D(s)$ and using the approximation of neglecting the one-loop contribution to the $N$ function, i.e., that of the $t$ and $u$ channels, one can set $N(s)=s/96\pi v^2$. Using the elastic unitarity relation, $\mathrm{Im}D(s)=-N(s)\rho(s)$ with $\rho(s)=1$ in the chiral limit, one has

$$D(s)=1+\beta s-\frac{s^2}{96\pi^2 v^2}\int_0^\infty ds'\frac{1}{s'(s'-s-i\varepsilon)}.$$

Defining $m_v^2$ as the position where the $P$-wave phase shift passes through 90° and absorbing the infrared divergence of the integral in the definition of $m_v^2$, one has

$$f_1(s)=\frac{s}{96\pi v^2}\frac{m_v^2}{m_v^2-s+(sm_v^2/96\pi^2v^2)\ln(-s/m_v^2)}, \tag{12}$$

where for $s\geq 0$, $\ln(-s)=\ln(s)-i\pi$. The appearance of the logarithm function in Eq. (12) is due to the $N/D$ method and the approximation of neglecting the NG boson loop contribution to the left-hand cut in Eq. (12) and was first given by Brown and Goble [11]. The vector meson width satisfies to a good accuracy the KSRF relation [6]. Assuming that $m_v^2=8\pi^2v^2$ as above, the expression for $f_1(s)$ has a pole on the real negative axis at $s=-1.6\times10^5 m_v^2$ and is 5 orders of magnitude larger than the physical scale of interest. This pole can be removed without the slightest influence on the low-energy calculation.

The problem with the $N/D$ method, as mentioned above, is that it is not possible to justify the neglect of the NG boson loop contribution to the $N$ function. The final result differs little, however, from that given by the Padé or the inverse amplitude method because the coefficient of the logarithm term in the $N/D$ method is small.

In conclusion, all the known unitarization methods, the Padé, the inverse amplitude, and the $N/D$, except the usual $K$-matrix method, as applied to the $P$-wave chiral perturbation theory, give rise to a $P$-wave resonance. Indeed, in a recent QCD calculation, using the Padé method and with the $O(p^4)$ term calculated from the fermion (quark) loop contribution with the number of colors $N_c=3$, a $\rho$ resonance was produced having the correct observed mass and width [12] ($m_v^2=24\pi^2v^2/N_c$). Using the same $O(p^4)$ term, the usual $K$-matrix solution does not give rise to the $\rho$ resonance but introduces instead two nearby poles in the physical sheet and hence must be rejected. It should be stressed again that, whatever unitarization scheme is used, it is important to check, after the calculation, the presence of nearby poles and cuts which could invalidate the unitarization method.

[a]Electronic address (bitnet): pthtnt@frpoly11.

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