Theoretical Considerations Concerning the Z^0 Mass

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We compare two previously introduced theoretical definitions of the Z^0 mass and show that one of them, frequently used, is gauge dependent in $O(\alpha^3)$ in a wide class of gauges and in $O(\alpha^2)$ in the complementary class. We then discuss a slightly modified version that circumvents these problems, is endowed with desirable theoretical properties, and can be identified with the mass measured at the CERN e^+e^- collider LEP. The simple formulation that emerges is applied to illustrate properties of the physical amplitude, related to gauge invariance, and the role played by the two definitions of mass.

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Over the last two years, the Z^0 mass has become one of the most precisely measured and important parameters in electroweak physics. Indeed, in conjunction with the Fermi constant G_μ and the fine-structure constant α, m_{Z^0} plays a crucial role in the verification of the standard model at the level of its quantum corrections, in the derivation of constraints on m_t and m_H, and in searches for new physics. A recent value, $m_{Z^0}=91.174\pm0.021$ GeV [1], bears witness to the accuracy reached. What is then this fundamental parameter, the Z^0 mass, from a theorist's perspective? That the issue is not trivial follows from the inherent ambiguities placed upon the mass of an unstable particle by the uncertainty principle [2]. And yet, in order to correlate different observables at the level of the quantum corrections, the theorist requires a precise theoretical concept. The riddle becomes altogether more acute when one recalls that the width of Z^0 is about 2.5 GeV, more than 100 times larger than the mass uncertainty.

We begin our analysis by comparing two previously introduced theoretical definitions of the Z^0 mass. In order to discuss this concept one considers the unrenormalized propagator

$$\Delta_{\mu\nu}(q)=-ig_{\mu\nu}/[q^2-m_0^2-A(q^2)]+\cdots, \tag{1}$$

where the ellipses stand for $q_\mu q_\nu$ terms, m_0^2 is the bare squared mass, and $A(q^2)$ represents the unrenormalized self-energy [it includes γZ mixing effects that start in $O(\alpha^2)$ and tadpole and tadpole counterterms that contribute only at $q^2=0$].

The most commonly used definition m_1 is

$$m_1^2=m_0^2+\mathrm{Re}A(m_1^2), \tag{2}$$

i.e., the zero of the real part of the denominator in Eq. (1). It corresponds to the usual field-theoretic treatment of stable particles.

An alternative, more fundamental definition, involves the complex-valued position of the pole in Eq. (1). Calling $s=q^2$, and $\bar{s}$ the pole position, we have

$$\bar{s}=m_0^2+A(\bar{s}). \tag{3}$$

In terms of $\bar{s}$ one defines the mass m_2 and the width Γ_2 by

$$\bar{s}=m_2^2-im_2\Gamma_2. \tag{4}$$

In the context of Z^0 physics the definition of Eqs. (3) and (4) was proposed some time ago by Consoli and the present author [3]. The idea of using Eq. (3) has been rediscovered recently by Willenbrock and Valencia [4], who employ a parametrization different from Eq. (4). An important feature of this "pole definition," stressed in Ref. [4], is that it is a basic property of the S matrix and one therefore expects it to be intrinsically gauge invariant.

In order to compare m_1 and m_2, we note that $\Gamma_2=O(\alpha)$ (α represents here a generic gauge coupling), expand Eq. (3) up to terms of $O(\alpha^3)$,

$$\bar{s}=m_0^2+A(m_2^2)-A'(m_2^2)im_2\Gamma_2 -A''(m_2^2)m_2^2\Gamma_2^2/2+\cdots, \tag{5}$$

and separate real and imaginary parts,

$$m_2^2=m_0^2+\mathrm{Re}A(m_2^2)+\mathrm{Im}A'(m_2^2)m_2\Gamma_2 -\mathrm{Re}A''(m_2^2)m_2^2\Gamma_2^2/2+\cdots, \tag{6}$$

$$m_2\Gamma_2=-\mathrm{Im}A(m_2^2)+\mathrm{Re}A'(m_2^2)m_2\Gamma_2 +\mathrm{Im}A''(m_2^2)m_2^2\Gamma_2^2/2+\cdots, \tag{7}$$

where henceforth the ellipses represent terms of $O(\alpha^4)$ and higher. Next, we write $\mathrm{Im}A'(m_2^2)=\mathrm{Im}A_1'(m_2^2)+\mathrm{Im}A_2'(m_2^2)$, where the subscripts 1 and 2 denote one- and two-loop contributions, respectively, and express Eq. (6) as

$$M^2=m_0^2+\mathrm{Re}A(M^2)+\mathrm{Re}A'(m_2^2)\mathrm{Im}A_1'(m_2^2)m_2\Gamma_2 +\mathrm{Im}A_2'(m_2^2)m_2\Gamma_2 -\mathrm{Re}A''(m_2^2)m_2^2\Gamma_2^2/2+\cdots, \tag{8}$$

$$M^2\equiv m_2^2-\mathrm{Im}A_1'(m_2^2)m_2\Gamma_2. \tag{9}$$

Subtracting Eq. (8) from Eq. (2) and using the mean-value theorem to estimate $\mathrm{Re}A(m_1^2)-\mathrm{Re}A(M^2)$, one finds

$$\Delta = -\mathrm{Im}A_2'(m_2^2)m_2\Gamma_2 - \mathrm{Re}A'(m_2^2)\mathrm{Im}A_1'(m_2^2)m_2\Gamma_2 + \mathrm{Re}A''(m_2^2)m_2^2\Gamma_2^2/2 + \cdots , \tag{10}$$

$$\Delta \equiv m_1^2 - M^2 = m_1^2 - m_2^2 + \mathrm{Im}A_1'(m_2^2)m_2\Gamma_2 . \tag{11}$$

The last term in Eq. (11) is of $O(\alpha^2)$ and gauge invariant in the class of gauges $\mathcal{C}_1$ defined by $\infty > \xi \geq (4\cos^2\theta_W)^{-1}$, but is gauge dependent in the complementary set $\mathcal{C}_2$ (here we use a convention where $\xi \to \infty$, $\xi = 1$, and $\xi = 0$ correspond to the unitary, 't Hooft-Feynman, and Landau gauges, respectively). This can be seen as follows: For values of ξ such that $4m_W^2\xi \geq m_2^2$, the bosonic contributions to $A_1(s)$ cannot develop an imaginary part in the neighborhood of $s = m_2^2$ because the unphysical bosonic excitations are too massive. Therefore, at $s \approx m_2^2$, $\mathrm{Im}A_1(s)$ and $\mathrm{Im}A_1'(s)$ involve only one-loop fermionic contributions which are gauge invariant. In $\mathcal{C}_2$ this is not the case: Although $\mathrm{Im}A_1(m_2^2)$, in conformity with Eq. (7), is gauge invariant, in the neighborhood of $s = m_2^2$, the amplitude $\mathrm{Im}A_1(s)$ contains bosonic contributions and $\mathrm{Im}A_1'(m_2^2)$ is gauge dependent. As the terms on the right-hand side (rhs) of Eq. (10) are of $O(\alpha^3)$, they cannot compensate for this difficulty and we conclude that in $\mathcal{C}_2$, m_1^2 and m_2^2 differ by gauge-dependent terms of $O(\alpha^2)$. Although $\mathcal{C}_2$ contains the Landau gauge, most calculations in electroweak physics are carried out in the 't Hooft-Feynman and unitary gauges, so that we now turn our attention to the $\mathcal{C}_1$ class. In this case the last term in Eq. (11) does not present a problem, but the three $O(\alpha^3)$ contributions on the rhs of Eq. (10) are gauge dependent. A possible caveat is that their gauge dependences may cancel. We now show that this is not the case. We note that in $\mathcal{C}_1$ and in the vicinity of $s = m_2^2$, $\mathrm{Im}A_2(s)$ is related to the one-loop vertex corrections to $\Gamma(s)$ (the width of a Z particle of squared mass s). These vertex corrections contain gauge-dependent parts proportional to the Z^0 and γ fermionic currents, J_Z and J_γ. The contributions of the former are proportional to the zeroth-order width $\Gamma^0(s) = -\mathrm{Im}A_1(s)/m_2$, so that altogether we can write

$$\mathrm{Im}A_2(s) = 2\,\mathrm{Im}A_1(s)\delta V(s) - m_2(\delta\Gamma)_{\gamma Z} + (\mathrm{g.i.}) , \tag{12}$$

where $\delta V(s)$ represents the one-loop gauge-dependent vertex corrections multiplying J_Z in the decay amplitude, $-m_2(\delta\Gamma)_{\gamma Z}$ are gauge-dependent contributions bilinear in matrix elements of J_γ and J_Z, and (g.i.) denotes gauge-independent terms. Including the contribution of the first term of Eq. (12), the rhs of Eq. (10) becomes

$$-m_2\Gamma_2\,\mathrm{Im}A_1'(m_2^2)[2\delta V(m_2^2) + \mathrm{Re}A'(m_2^2)] + m_2^2\Gamma_2^2[2\delta V'(m_2^2) + \mathrm{Re}A''(m_2^2)/2] + \cdots .$$

The first term is gauge invariant since the combination $2\delta V(m_2^2) + \mathrm{Re}A'(m_2^2)$ is precisely what appears in the $O(\alpha)$ correction to the width. However, the second term, involving $2\delta V'(m_2^2) + \mathrm{Re}A''(m_2^2)/2$, is not. This can be verified mathematically, but can also be understood by a physical argument. As we will see later, when one considers the $O(\alpha)$ corrections to a four-fermion process such as $e^+e^- \to f\bar{f}$ at $s = m_2^2$, there is a "nonresonant part" that includes precisely the terms above plus additional gauge-dependent contributions from the box diagrams. The gauge dependences cancel in the physical amplitude, but not in $2\delta V'(m_2^2) + \mathrm{Re}A''(m_2^2)/2$, since it represents only part of the nonresonant contribution. We conclude that the rhs of Eq. (10) is gauge dependent. Furthermore, if in $\mathrm{Im}A_1(s)$ we neglect small gauge-invariant corrections of $O(\alpha m_b^2/m_Z^2)$ (scaling approximation), we can write in Eq. (11)

$$\mathrm{Im}A_1'(m_2^2)m_2\Gamma_2 \approx \mathrm{Im}A_1(m_2^2)\Gamma_2/m_2 = -\Gamma_2^2 + O(\alpha^3) .$$

In summary, in $\mathcal{C}_1$ Eq. (10) becomes

$$m_1^2 = m_2^2 + \Gamma_2^2 + O(\alpha^3) , \tag{13}$$

where some of the terms of $O(\alpha^3)$ are gauge dependent. Returning to $\mathcal{C}_2$ we can separate in Eq. (11)

$$\mathrm{Im}A_1'(m_2^2)m_2\Gamma_2 = \{[\mathrm{Im}A_1'(m_2^2)]_f + [\mathrm{Im}A_1'(m_2^2)]_b\}m_2\Gamma_2 ,$$

where the subscripts f and b denote the fermionic and bosonic parts. We write again the first term as $-\Gamma_2^2$ and, combining Eq. (10) and Eq. (11), obtain in $\mathcal{C}_2$

$$m_1^2 = m_2^2 + \Gamma_2^2 - [\mathrm{Im}A_1'(m_2^2)]_b m_2\Gamma_2 + O(\alpha^3) . \tag{14}$$

As emphasized before, the third term on the rhs is of $O(\alpha^2)$ and gauge dependent. Because it only arises in the restricted class $\xi \leq (4\cos^2\theta_W)^{-1}$, far away from the unitary gauge, it is expected to be bounded in magnitude and small.

In order to obtain a gauge-invariant definition, we must somehow relax Eq. (2). A simple possibility, suggested by Eq. (13), is to replace Eq. (2) by

$$m_1^2 \equiv m_2^2 + \Gamma_2^2 , \tag{15}$$

which henceforth is regarded as exact. We note that m_1 is larger than m_2 by ≈ 34 MeV, i.e., 1.6 times the current experimental error. In terms of this modified m_1, Eq. (2) is altered by terms of $O(\alpha^3)$ in $\mathcal{C}_1$ and $O(\alpha^2)$ in $\mathcal{C}_2$. This means that the mass-renormalization counterterm $\delta m_1^2 = m_1^2 - m_0^2$ is not given exactly by $\mathrm{Re}A(m_1^2)$, but it is corrected by gauge-dependent contributions starting at those orders. We also introduce a slightly modified width,

$$\Gamma_1 = m_1\Gamma_2/m_2 . \tag{16}$$

With this definition, regarded as exact, $m_1\Gamma_1$ satisfies an expression analogous to Eq. (7) with $m_2^2 \to m_1^2$.

In order to understand more clearly the meaning of m_1 and Γ_1 and the connection with physics observed at the CERN e^+e^- collider LEP, one combines Eq. (1) and

Eq. (3), i.e.,

$$s-m_0^2-A(s)=s-\bar{s}-[A(s)-A(\bar{s})]\,, \tag{17}$$

and, following Ref. [3], separates the propagator into "resonant" and "nonresonant" parts,

$$\frac{1}{s-m_0^2-A(s)}=\frac{1+(s-\bar{s})f(s)+\cdots}{(s-\bar{s})[1-A'(\bar{s})]}\,, \tag{18}$$

$$f(s)=\frac{A(s)-A(\bar{s})-A'(\bar{s})(s-\bar{s})}{(s-\bar{s})^2[1-A'(\bar{s})]}\,, \tag{19}$$

where the ellipses here denote higher powers of $(s-\bar{s})f(s)$. The function $f(s)$ is regular as $s\to\bar{s}$ so that terms involving $f(s)$ represent nonresonant contributions. Equations (17) and (18) also reveal that in a perturbative expansion based on $\bar{s}$, $A(\bar{s})$ and $[1-A'(\bar{s})]^{-1}$, complex-valued quantities, would play the role of "mass counterterm" and "field renormalization constant," respectively. Multiplying and dividing by $1+i\Gamma_2/m_2$ and recalling Eqs. (15) and (16), the pole term in Eq. (18) can be written in an equivalent form,

$$\text{pole}=(s-m_1^2+is\Gamma_1/m_1)^{-1}[1-A'(\bar{s})]^{-1}(1+i\Gamma_1/m_1)\,. \tag{20}$$

If we restrict ourselves for simplicity to $\mathcal{C}_1$ and if in the second factor we neglect terms of $O(\alpha^3)$ in the real parts and $O(\alpha^2)$ in some imaginary parts, Eq. (20) simplifies to

$$\text{pole}=(s-m_1^2+is\Gamma_1/m_1)^{-1}[1-\text{Re}A'(m_1^2)+\cdots]^{-1}\,. \tag{21}$$

Equation (21) exhibits the characteristic Breit-Wigner form, with s-dependent width and real field renormalization, employed in current analyses of LEP physics. Thus, m_1 and Γ_1 can be identified with the corresponding quantities measured in those experiments. Equation (21) conforms also with one's intuitive feeling of how the pole term should behave: a Breit-Wigner form near resonance with a much smaller imaginary part for $s\ll m_1^2$. The latter property is of course present in the corresponding pole term in Eq. (18), but there the cancellation of imaginary parts at low s is less explicit. Indeed, it is easy to see starting with Eq. (1), that the mass parameter that naturally appears in calculations at $s<0$ or at $s\ll m_1^2$, where the $O(\alpha)$ imaginary parts are very small, is m_1 rather than m_2. Finally, the modified parameter m_1 conforms with the traditional renormalization condition of Eq. (2) through terms of $O(\alpha^2)$ ($\mathcal{C}_1$) and $O(\alpha)$ ($\mathcal{C}_2$). Thus, although we regard the complex-pole definition [Eqs. (3) and (4)] as the basic one, we see that Eqs. (15) and (16) provide an alternative description in terms of the parameters m_1 and Γ_1, endowed with desirable theoretical properties.

The transformations of Eqs. (15) and (16) have been extensively used before [5], but in a different theoretical context. The traditional approach has been to start with Eq. (2), derive a pole term of the form of Eq. (21), and apply the inverse transformations in order to obtain a description in terms of an s-independent width, as this is convenient for practical purposes. In some sense our approach has been the opposite. After demonstrating the theoretical difficulties associated with Eq. (2) in high orders of perturbation theory, we have proposed to start with Eqs. (3) and (4) as the basic definition and used Eqs. (15) and (16) to introduce an alternative description in terms of m_1 and Γ_1.

The decomposition into pole terms and nonresonating parts can be used to discuss properties of the physical amplitude related to gauge invariance. For example, consider a four-fermion process and call $V_1(s)$ and $V_2(s)$ the vertex parts multiplying $\langle 1'|J_\mu^Z|1\rangle$ and $\langle 2'|J_\mu^Z|2\rangle$, respectively. Writing $V_i(s)=V_i(\bar{s})+(s-\bar{s})g_i(s)$ $(i=1,2)$, one notices that the residue of the pole is given by $V_1(\bar{s})V_2(\bar{s})[1-A_Z'(\bar{s})]^{-1}$ and must be gauge invariant. Expanding the residue in powers of α one gains information about gauge-invariant combinations of vertex parts, self-energies, and their derivatives to various orders in α. One can also use this decomposition to demonstrate explicitly the gauge invariance of the $O(\alpha)$ corrections for arbitrary values of s, which is complicated to some extent by the resummation implied by the resonance [6]. One notices that to this order the residue of the pole is $V_1(m_1^2)+V_2(m_1^2)+A_Z'(m_1^2)$. The nonresonant contributions are given by $g_1(s)+g_2(s)+f(s)+b(s,t)$, where $b(s,t)$ symbolically represents the box diagrams (t is a second Mandelstam variable); as these functions are already of $O(\alpha)$ and regular as $s\to\bar{s}$, we can further approximate $\bar{s}\approx m_1^2$. We have indeed verified that these two sets of contributions are gauge independent to $O(\alpha)$ for all values of s.

Finally, we would like to point out that the considerations of this paper can be extended in a natural manner to other unstable particles.

After completing this work, the author's attention has been called to a recent preprint by Stuart [7], where the complex pole definition is also discussed. Some of Stuart's conclusions are equivalent to the observations made in the penultimate paragraph of the present paper.

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