Quasilocal Mass Constructions with Positive Energy

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We propose a pair of new quasilocal constructions for the four-momentum and mass of the gravitational and matter fields within generic two-surfaces. We show that the momenta are future pointing when the dominant energy condition is satisfied on a spanning three-surface and the two-surface is suitably convex. The new definition gives zero in flat space and the correct results in linearized theory, at spacelike infinity and for round spheres. At null infinity at least one definition gives the Bondi mass. These definitions can be embedded in definitions of angular momentum.

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The purpose of this Letter is to present a pair of definitions of the momentum and mass of the gravitational and matter fields contained within generic two-spheres in the context of general relativity. The existence and meaning of such definitions has always been contentious —translation symmetries are usually required for the definition of energy-momentum, whereas such symmetries do not exist in general curved space-times. Nevertheless, there has been much interest in the possibility that such definitions could have applications to the interpretation of numerical and exact solutions of Einstein's field equations, provide useful techniques for use by those interested in analytic results for general space-times, and form a significant piece of general theory. We resolve the difficulty by observing that one only needs to define translations at the bounding two-surface $\mathcal{S}$ of the region whose mass, etc., one is interested in computing. This turns out to be feasible. It is perhaps worth emphasizing that a crucial property of such a definition is that of positivity when the space-time satisfies the dominant energy condition.

We first motivate and state the new definitions. We then show that the definitions give the correct answer in flat-space linearized theory and at spacelike infinity. We go on to prove a positivity theorem. Finally, we survey the results of calculations of the mass definition for large spheres near null infinity, for small spheres, and for round spheres.

The new definitions are a substantial improvement on previous ones. The definition given by Hawking [1] fails to give zero for general two-surfaces in flat space-time. The definition of Ludvigsen and Vickers [2] is not genuinely quasilocal since it requires that the two-surface be attached to null infinity (or else to a point) by a null hypersurface. A quasilocal formulation should only depend on the intrinsic and extrinsic geometry of the two-surface in question. The definition of Bartnik [3] is, by definition, optimal with respect to its defining criterion but is not usable in practice. Penrose's definition [4] is the most serious contender, but as yet gives no definitive way of calculating the mass within a general two-surface and there is no definition of energy momentum in general. Furthermore, a positivity theorem is not yet available even where the mass definition is unambiguous (see Tod [5]).

The positivity theorem and the success of the new definitions in other regimes means that they are major contenders for a four-momentum and mass definition for general two-surfaces.

Motivation and definition.— The point of view that we shall adopt is that a component of the (angular) momentum is the value of the Hamiltonian that generates the corresponding translation (rotation). There are various difficulties with implementing this point of view rigorously in the case of gravitation so the motivation for the formula is partly heuristic. See [6] for further discussion.

The total Hamiltonian: We have the Sen-Witten identity (see for instance [7])

$$d(i\lambda_{A'}D\lambda_A\wedge\theta^{AA'})=iD\lambda_{A'}\wedge D\lambda_A\wedge\theta^{AA'}+\tfrac{1}{2}\lambda^A\lambda^{A'}G_{AA'b}\varepsilon^b{}_{cde}\theta^c\wedge\theta^d\wedge\theta^e$$

or in Dirac spinor notation

$$d(\bar\psi\gamma\theta^a\gamma_a\wedge D\psi)=D\bar\psi\wedge\gamma\theta^a\gamma_a\wedge D\psi+\tfrac{1}{2}G^{ab}(\bar\psi\gamma_a\psi)\varepsilon_{bcde}\theta^c\wedge\theta^d\wedge\theta^3 .$$

Here the θ^a are a tetrad of one-forms, the γ_a are the Clifford algebra matrices, $\gamma=\frac{1}{24}\varepsilon^{abcd}\gamma_a\gamma_b\gamma_c\gamma_d$, G_{ab} is the Einstein tensor, D is the covariant exterior derivative, ψ is a Dirac spinor, and λ_A is a two-component Weyl spinor. The first term on the right-hand side is the integrand for the gravitational Hamiltonian, and, when the field equations hold, the second term on the right-hand side is the integrand for the matter Hamiltonian that generates motions along the vector field $\lambda_{A'}\lambda_A=\bar\psi\gamma_a\psi$. The left-hand side is therefore the integrand for the total Hamiltonian when the field equations are satisfied.

The Hamiltonian has the following properties. (1) The total Hamiltonian is an exact form when the field equations are satisfied so that the integral over some finite piece of three-surface Σ depends only on the spinors and data on the bounding two-surface $\mathcal{S}=\partial\Sigma$. This rather general fact leads to the weak conservation [4] of momenta in general relativity, that is, the independence of the momenta from the choice of spanning three-surface.

(2) The value of the Hamiltonian depends not only on the vector $\lambda_A\lambda_{A'}$, but also on its decomposition into spinors. For a definition of momenta we therefore need not only a "quasilocal" definition of Killing vectors at $\mathcal{S}$, but also a canonical decomposition of these "quasi Killing vectors" into products of spinors.

The quasitranslations and corresponding momenta: In order to define a real four-momentum we must have a definition of real "quasitranslations" together with their decomposition into spinors at $\mathcal{S}$.

The new definitions use spinors $\lambda_{A'}$ and their complex conjugates that are holomorphic or antiholomorphic on $\mathcal{S}$ relative to the complex structure induced on $\mathcal{S}$ by the metric and on the spin bundle $\mathbf{S}_{A'}$ by the spin connection. (This type of idea is used in the context of Yang-Mills theory in the definition of quasilocal charges for Yang-Mills by Tod [9].) There are therefore two definitions, one where the primed spinors are holomorphic, and the other where they are antiholomorphic. This leads to the following equations:

$$\bar{\delta}\lambda_{B'} = \iota^A o^{A'}\nabla_{AA'}\lambda_{B'} = 0$$

and

$$\delta\lambda_{B'} = o^A \iota^{A'}\nabla_{AA'}\lambda_{B'} = 0 ,$$

where $o^{A'}o^A$ and $\iota^A\iota^{A'}$ are, respectively, the outward and inward null normals to $\mathcal{S}$ and $\nabla_{AA'}$ is the space-time spin connection ($\bar{\delta}$ is the $\bar{\eth}$ operator on $\mathbf{S}_{A'}|_{\mathcal{S}}$). For brevity we shall concentrate mainly on the holomorphic case—the other case is obtained by time reversal of all the formulas.

Generically $\mathbf{S}_{A'}$ is trivial as a holomorphic vector bundle (with respect to either δ or $\bar{\delta}$) on $\mathcal{S}$ and so there exists precisely two linearly independent solutions $(\lambda^{0'}_{A'}, \lambda^{1'}_{A'}) = \lambda^{\underline{A}'}_{A'}$, $\underline{A}' = 0', 1'$. Note that there will, however, be exceptional two-surfaces on which $\mathbf{S}_{A'}$ is not trivial as a holomorphic vector bundle. When this happens the solutions will be proportional and, perhaps in very exceptional cases, there will be more than two solutions. For such two-surfaces the construction as it stands breaks down.

We now define a quasitranslation to be a four-vector field on $\mathcal{S}$ of the form

$$K_{AA'} = K_{\underline{AA}'}\lambda^{\underline{A}'}_{A'}\bar{\lambda}^{\underline{A}}_{A} ,$$

where the $K_{\underline{AA}'}$ are constants. Quasitranslations are thus given by linear combinations of these spinors and their complex conjugates. The resulting value of the Hamiltonian for $K_{AA'}$ is $P^{\underline{AA}'}K_{\underline{AA}'}$, where the momentum vector $P^{\underline{AA}'}$ is defined to be

$$P^{\underline{AA}'} = \frac{1}{4\pi G}\oint_{\mathcal{S}} -i\lambda^{\underline{A}'}_{A'} D\lambda^{\underline{A}}_{A}\wedge\theta^{AA'} .$$

(Penrose's definition can be understood within this framework, but the definition provides ten complex quasi Killing vectors with, in general, no method to determine which of them are translations or any guarantee that any of them are real; see [8] and [6].)

The mass: In order to define a mass, we must be able to define a constant $\varepsilon_{\underline{AB}}$ so that we can define

$$m^2 = P^{\underline{AA}'}P^{\underline{BB}'}\varepsilon_{\underline{AB}}\varepsilon_{\underline{A}'\underline{B}'} .$$

The natural definition is $\varepsilon^{\underline{A}'\underline{B}'} = \lambda^{\underline{A}'}_{A'}\lambda^{\underline{B}'}_{B'}\varepsilon^{A'B'}$. It follows from $\bar{\delta}\lambda^{\underline{A}'}_{A'} = 0$ that $\bar{\delta}\varepsilon^{\underline{A}'\underline{B}'} = 0$, so that the $\varepsilon^{\underline{A}'\underline{B}'}$ are holomorphic and global functions on the sphere and hence, by Liouville's theorem, constant.

Angular momentum: One can define more general quasi Killing vectors that include rotations using local twistors $(\omega^A, \pi_{A'})$ restricted to $\mathcal{S}$ satisfying either $\bar{\delta}(\omega^A, \pi_{A'}) = 0$ or $\delta(\omega^A, \pi_{A'}) = 0$, where δ and $\bar{\delta}$ are the holomorphic and antiholomorphic vector fields on $\mathcal{S}$ as above acting according to the local twistor connection. These equations are guaranteed to have just four independent solutions generically since as before these are $\bar{\eth}$-type equations whose solutions are holomorphic sections of a holomorphic vector bundle on the sphere $\mathcal{S}$. Generically the holomorphic vector bundle will be trivial and so there will be just four linearly independent solutions $(\omega^A_\alpha, \pi_{\alpha A'}) = ((\omega^A_0, \pi_{0,A'}), \ldots, (\omega^A_3, \pi_{3A'}))$. These can be used to define quasi Killing vectors, and quasiconformal Killing vectors according to various recipes (see for instance [6]) which then give rise to angular momenta by substitution into the Witten-Nester form.

Flat-space linearized theory and infinity.—In flat space, the $\lambda_{A'}$ given by both definitions are guaranteed to be the restriction to $\mathcal{S}$ of the constant spinors, since they certainly satisfy the δ and the $\bar{\delta}$ equation, and the solutions are unique. The integrand therefore vanishes since it contains the derivative of the constant $\lambda_{A'}$ giving the correct answer $P_{\underline{AA}'} = 0$.

A similar argument works at spacelike infinity. The asymptotically constant spinors certainly satisfy both the δ and $\bar{\delta}$ equation asymptotically, and so, by uniqueness, span the solution spaces. The integral is then the expression for the Arnowitt-Deser-Misner energy as used by Witten [10]; see also [11].

In order to see that one gets the correct answer for linearized theory, it is convenient to turn the integral into a volume integral using the "Sen-Witten" identity:

$$\frac{1}{4\pi G}\oint_{\mathcal{S}} -i\lambda^{\underline{A}'}_{A'} D\lambda^{\underline{A}}_{A}\wedge\theta^{AA'} = \frac{1}{4\pi G}\int_{\Sigma}\{-iD\lambda^{\underline{A}'}_{A'}\wedge D\lambda^{\underline{A}}_{A}\wedge\theta^{AA'} - \tfrac{1}{2}\lambda^{\underline{A}}_{A}\lambda^{\underline{A}'}_{A'}G^{AA'b}\varepsilon_{bcde}\theta^c\wedge\theta^d\wedge\theta^e\} ,$$

where Σ is some three-surface that spans $\mathcal{S}$. Since to zeroth order the $\lambda_{A'}$'s are constant, $D\lambda_{A'}$ are first order so that the first term on the right-hand side is of order ε^2. The linearized Einstein tensor G_{ab} is already of order ε so that the $\lambda_{A'}$'s in that term can be taken to be the constant spinors. This second term therefore gives the correct answer, that is, the integral of the momentum density of the source. (Energy-momentum is part of the charge integral for general relativity

[4].)

Positivity.— For a good definition it is essential that the momentum vector should be future pointing when the two-surface can be spanned by a three-surface on which the data satisfy the dominant energy condition. The following argument is an adaptation of a positivity argument used in a different context in Ludvigsen and Vickers [2] and based on [10]. In the following we show that $P^{00'}$ is non-negative, and write, for simplicity, $\lambda_{A'}=\lambda^{0'}_{A'}$.

Theorem: The quasilocal momentum component $P^{00'}$ defined by $\bar\delta\lambda_{A'}=0$ ($\delta\lambda_{A'}=0$) is non-negative whenever $\rho'\geq 0$ ($\rho\leq 0$) and $\mathcal{S}$ is spanned by a three-surface on which the dominant energy condition is satisfied.

Proof: Let $\lambda_{A'}$ be some field defined on a two-surface $\mathcal{S}$ and let $I_\lambda(\mathcal{S})$ be the integral of the Witten-Nester two-form $\Lambda=i\bar\lambda_A D\lambda_{A'}\wedge\theta^{AA'}$ over $\mathcal{S}$. In spin coefficients and the Geroch-Held-Penrose formalism [7,12], this may be written as

$$I_\lambda(\mathcal{S})=\oint_{\mathcal{S}}\{\bar\lambda_1(\eth\lambda_{0'}+\rho\lambda_{1'})-\bar\lambda_0(\bar\eth\lambda_{1'}+\rho'\lambda_{0'})\}dS\,. \tag{1}$$

Consider first the system of equations $\bar\delta\lambda_{A'}=0$:

$$\bar\eth\lambda_{1'}+\rho'\lambda_{0'}=0\,, \tag{2}$$

$$\bar\eth\lambda_{0'}+\bar\sigma\lambda_{1'}=0\,. \tag{3}$$

Then using (2) and integrating by parts we get

$$4\pi G P^{00'}=I_\lambda(\mathcal{S})=\oint_{\mathcal{S}}(\rho'\bar\lambda_0\lambda_{0'}+\rho\bar\lambda_1\lambda_{1'})dS\,. \tag{4}$$

Let $\mathcal{S}$ be spanned by a nonsingular spacelike three-surface Σ on which the dominant energy condition is satisfied. The Sen-Witten equation $n^{A(A'}\nabla^{B')}_{A}\tilde\lambda_{A'}=0$ on Σ (where n^a is a normal to Σ) is an elliptic system of two first-order partial differential equations. We may therefore find a solution [13] $\tilde\lambda_{A'}$ on Σ satisfying the boundary condition

$$\tilde\lambda_{1'}=\lambda_{1'} \tag{5}$$

on $\mathcal{S}$. In general, $\tilde\lambda_{0'}$ will differ from $\lambda_{0'}$ on $\mathcal{S}$. Denote this difference by

$$Y=\tilde\lambda_{0'}-\lambda_{0'}\,. \tag{6}$$

We now relate $I_\lambda(\mathcal{S})$ to $I_{\tilde\lambda}(\mathcal{S})$:

$$\begin{aligned}
I_{\tilde\lambda}(\mathcal{S})&=\oint_{\mathcal{S}}\{\bar{\tilde\lambda}_1(\eth\tilde\lambda_{0'}+\rho\tilde\lambda_{1'})-\bar{\tilde\lambda}_0(\bar\eth\tilde\lambda_{1'}+\rho'\tilde\lambda_{0'})\}dS\\
&=\oint_{\mathcal{S}}\{\bar\lambda_1(\eth\tilde\lambda_{0'}+\rho\lambda_{1'})-\bar{\tilde\lambda}_0(\bar\eth\lambda_{1'}+\rho'\tilde\lambda_{0'})\}dS\\
&=\oint_{\mathcal{S}}\{\rho'\bar\lambda_0\tilde\lambda_{0'}+\rho\bar\lambda_1\lambda_{1'}+\rho'\bar{\tilde\lambda}_0\lambda_{0'}-\rho'\bar{\tilde\lambda}_0\tilde\lambda_{0'}\}dS\\
&=\oint_{\mathcal{S}}\{\rho'\bar\lambda_0\lambda_{0'}+\rho\bar\lambda_1\lambda_{1'}-\rho'(\bar{\tilde\lambda}_0-\bar\lambda_0)(\tilde\lambda_{0'}-\lambda_{0'})\}dS\\
&=I_\lambda(\mathcal{S})-\oint_{\mathcal{S}}\rho' Y\bar Y\,dS\,,
\end{aligned}$$

where we have used Eqs. (2), (4), (5), and (6) and an integration by parts. When the matter on Σ satisfies the dominant energy condition we have the inequality $I_{\tilde\lambda}(\mathcal{S})\geq 0$ [10]. Hence whenever $\rho'\geq 0$, $I_\lambda(\mathcal{S})\geq 0$. Since $P^{00'}=(1/4\pi G)I_\lambda(\mathcal{S})$, this implies that $P^{\underline{AA'}}$ is future pointing as required.

Consider next the equation $\delta\lambda_{A'}=0$. An analogous argument to the one above but now with $\tilde\lambda_{0'}=\lambda_{0'}$ as boundary conditions for the Sen-Witten equation will show positivity whenever $\rho\leq 0$.

Since ρ and ρ' are the convergences of the outward and inward null normals to $\mathcal{S}$, the conditions $\rho\leq 0$ and $\rho'\geq 0$ are the condition that the two-surface is (suitably) convex, i.e., that there should be no indentations. This will be satisfied by a wide class of two-surfaces in a generic space-time.

Further results.— These momentum definitions have been computed for round spheres, for small spheres, and for large spheres that approach cuts of null infinity.

For round spheres both definitions give the "standard" answer $m=R^3(\Phi_{11}+\Lambda-\Psi_2)/G$ (where R is the radius as computed from the area, and the terms in parentheses are components of the curvature as in [7]). This is the same answer as Penrose's definition and the Hawking definition, see [14]. For small spheres both definitions give the expected contribution from the energy-momentum tensor at third order, and, when this vanishes, a contribution from the Bel-Robinson tensor at fifth order. At sixth order the two definitions differ.

For large spheres approximating a cut of future null infinity the definition for which the primed spinors are holomorphic gives the Bondi mass. In the absence of radiation the mass has been computed to third order in $1/r$, where r is an affine parameter along the null geodesics of an outgoing null hypersurface and the computations give reasonable answers. The definition for which the unprimed spinors are holomorphic diverges linearly in the affine parameter as the large sphere goes out to infinity when there is radiation. When there is no radiation this second definition gives the Bondi mass also. The roles of the two definitions are reversed at past null infinity.

These results will be given in full detail in a subsequent paper.

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[13] This is in fact rather a delicate point as this is a situation where the Fredholm alternative holds—one can only obtain a solution for general boundary data when the adjoint operator with the adjoint boundary condition has trivial kernel. However, with the assumptions of the theorem, $\rho' \geq 0$ and dominant energy in the interior, we have no solutions to the adjoint problem as this is the Witten equation with boundary condition $\lambda_1 = 0$ and then the Sen-Witten identity is then positive on one side and negative on the other forcing the solution to vanish. We are grateful to Paul Tod for obtaining the correct argument in this context.

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