PHYSICAL REVIEW LETTERS

VOLUME 67 | 14 OCTOBER 1991 | NUMBER 16

Classical and Quantum Superdiffusion in a Time-Dependent Random Potential

Leonardo Golubović,[a] Shechao Feng, and Fan-An Zeng

Department of Physics, University of California at Los Angeles, Los Angeles, California 90024

(Received 1 July 1991)

We consider wandering of a nonrelativistic particle in a time-dependent random potential in d spatial dimensions. Its root-mean-square displacement from the initial position increases superdiffusively with time t as $t^{9/8}$ for $d>1$, and as $t^{6/5}$ in $d=1$. Its kinetic energy increases as $t^{1/2}$ for $d>1$, and as $t^{2/5}$ in $d=1$. These scaling behaviors hold for both the classical and the corresponding quantum-mechanical problem in *continuous* space-time and differ from those of lattice models.

PACS numbers: 05.40.+j, 42.20.−y, 71.55.Jv

Quantum mechanics of a particle in a time-independent random potential has attracted enormous attention in the past years. On the other hand, interest in the dynamics in a *time-dependent* random potential started to grow only recently [1–4], though the problem was addressed long ago by Ovchinikov and Erikhman [5] and by others [6,7]. At the classical level this problem is described by Newton's equations

$$\frac{d\mathbf{p}}{dt}=-\frac{\partial U}{\partial \mathbf{x}},\quad \frac{d\mathbf{x}}{dt}=\frac{\mathbf{p}}{m}, \tag{1}$$

for the momentum $\mathbf{p}$ and the position $\mathbf{x}$ of a particle of the mass m. $U(\mathbf{x},t)$ is a time-dependent random potential, with zero mean and with short-range correlations *both* in time and in space, of the form

$$\langle U(\mathbf{x},t)U(\mathbf{x}',t')\rangle=G(|\mathbf{x}-\mathbf{x}'|,t-t'), \tag{2}$$

with $G(|\mathbf{x}|,t)$ rapidly falling off to zero for $|\mathbf{x}|>\xi_x$ or $|t|>\xi_t$, for example, $G\sim\exp(-\mathbf{x}^2/2\xi_x^2-t^2/2\xi_t^2)$. The corresponding quantum-mechanical problem described by the time-dependent Schrödinger equation,

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi+U(\mathbf{x},t)\psi, \tag{3}$$

recently attracted attention in relation to the propagation of directed waves in $(d+1)$-dimensional highly anisotropic scattering media [1]. Similar in form, though in detail very different, is the random directed polymer problem [8], which is the imaginary-time version of (3). The original, real-time model is interesting in its own right [5–7], for example, to model the motion of a light test particle placed in a gas of much heavier particles. If the test particle is at rest at $t=0$, its velocity will start to *grow* due to collisions with heavy particles. In a long range of time scales in which its velocity is still relatively small, one can neglect its influence on heavy particles and model their effect on the test particle by a time-dependent random potential. This model breaks down at sufficiently large t: The light particle eventually comes into equilibrium with heavy ones—its average velocity is constant while its wandering is ordinary diffusion at large t. Nonetheless, for time scales in which the *stricto sensu* random potential model is *still* appropriate, the test-particle velocity typically increases with time. Thus, its wandering, measured by the mean-square displacement from the initial position $\langle\mathbf{x}^2\rangle$, cannot be a simple diffusion. Since the particle, on average, *gains* the energy from the random time-dependent medium, one might expect a *superdiffusive* behavior [1,7] even for $t\to\infty$.

This Letter presents the first deep theoretical insight into superdiffusion of a *stricto sensu* time-dependent random potential model in d spatial dimensions. We find, for *both* the classical and quantum cases, that the scaling laws of this superdiffusion are *superuniversal* for any $d>1$: $\langle\mathbf{x}^2\rangle^{1/2}$ increases in a *superlinear* way as $t^{9/8}$ while the particle average kinetic energy $\langle E_{\rm kin}\rangle$ increases as $t^{1/2}$. The case $d=1$ is special. We find $\langle\mathbf{x}^2\rangle^{1/2}\sim t^{6/5}$, while $\langle E_{\rm kin}\rangle\sim t^{2/5}$. These behaviors, obtained for the *continuous* space-time model, are strikingly different from those of the lattice versions of (3) for which $\langle\mathbf{x}^2\rangle$ increases as in ordinary diffusion [2,4–6].

To derive analytically these conclusions for the classical problem, Eqs. (1) and (2), we will presume that $U(\mathbf{x},t)$ is a Gaussian random variable. By applying the classical Martin-Siggia-Rose formalism [9] to this problem, one directly obtains the probability weight of the particle path $\{\mathbf{x}(t)\}$ as the functional integral over an auxiliary vector function $\mathbf{Y}(t)=[Y_1(t),Y_2(t),\ldots,Y_d(t)]$:

$$P(\{\mathbf{x}(t)\})\sim\int D\mathbf{Y}\left\langle\exp\left[i\int dt\,\mathbf{Y}(t)\left(m\frac{d^2\mathbf{x}}{dt^2}+\frac{\partial U}{\partial\mathbf{x}}(\mathbf{x}(t),t)\right)\right]\right\rangle$$
$$\sim\int D\mathbf{Y}\exp\left[i\int dt\,mY_j\frac{d^2x_j}{dt^2}-\frac{1}{2}\int dt_1\int dt_2\,Y_j(t_1)F_{jk}(\mathbf{x}(t_1)-\mathbf{x}(t_2),t_1-t_2)Y_k(t_2)\right],\qquad(4)$$

where

$$F_{jk}(\mathbf{x},t)=-\frac{\partial^2}{\partial x_j\,\partial x_k}G(|\mathbf{x}|,t)$$

(summation over repeated indices is presumed). To discuss Eq. (4) we consider the weak disorder limit in which one can expand

$$\mathbf{x}(t_1)-\mathbf{x}(t_2)=(t_1-t_2)\left(\frac{d\mathbf{x}}{dt}\right)_{t=t_1}+\cdots,$$

and $Y_k(t_2)=Y_k(t_1)+\cdots$. Thus, to the leading order,

$$P(\{\mathbf{x}(t)\})\sim\int D\mathbf{Y}\exp\left[i\int dt\,mY_k(t)\frac{d^2x_k(t)}{dt^2}-\frac{1}{2}\int dt\,Y_j(t)M_{jk}\left(\frac{d\mathbf{x}}{dt}\right)Y_k(t)\right].\qquad(5)$$

Here the matrix M_{jk} is a function of the particle velocity $\mathbf{v}=d\mathbf{x}/dt$ given by

$$M_{jk}(\mathbf{v})=\int_{-\infty}^{\infty}dt\,[F_{jk}(\mathbf{x},t)]_{\mathbf{x}=t\mathbf{v}}$$
$$=M_L(v)\frac{v_jv_k}{v^2}+M_T(v)\left[\delta_{jk}-\frac{v_jv_k}{v^2}\right],\qquad(6)$$

with $v=|\mathbf{v}|$,

$$M_L(v)=-\int_{-\infty}^{\infty}dt\left(\frac{\partial^2G}{\partial x^2}(x,t)\right)_{x=tv},$$

and

$$M_T(v)=-\int_{-\infty}^{\infty}dt\left(\frac{1}{x}\frac{\partial G}{\partial x}(x,t)\right)_{x=tv}.$$

Equation (5) is exactly the Martin-Siggia-Rose generating functional for the following Langevin process describing the diffusion in the particle *velocity* space:

$$m\frac{dv_j}{dt}=[M^{1/2}(\mathbf{v})]_{jk}\eta_k$$
$$=\left[[M_L(v)]^{1/2}\frac{v_jv_k}{v^2}+[M_T(v)]^{1/2}\left(\delta_{jk}-\frac{v_jv_k}{v^2}\right)\right]\eta_k,\qquad(7)$$

where $\boldsymbol{\eta}$ is a Gaussian white noise with the correlator $\langle\eta_j(t)\eta_k(t')\rangle=\delta_{jk}\delta(t-t')$. The Fokker-Planck equation for the velocity distribution function $P(\mathbf{v},t)$ then reads

$$\frac{\partial P}{\partial t}=\frac{1}{m^2}\frac{\partial}{\partial v_j}[M^{1/2}(\mathbf{v})]_{jl}\frac{\partial}{\partial v_k}[M^{1/2}(\mathbf{v})]_{kl}P.\qquad(8)$$

For concreteness, let us consider $G(|\mathbf{x}|,t)=A\exp(-\mathbf{x}^2/2\xi_x^2-t^2/2\xi_t^2)$ for which $M_T=B[1+(v/v_0)^2]^{-1/2}$ and $M_L=B[1+(v/v_0)^2]^{-3/2}$, where $v_0=\xi_x/\xi_t$ is, in the following, an important velocity scale and $B=(2\pi)^{1/2}A\xi_t/\xi_x^2$. Note that for $v\ll v_0$, $M_T(v)\approx M_L(v)\approx B=$const, while for $v\gg v_0$, $M_T\sim v^{-1}$, $M_L\sim v^{-3}$. In fact, these asymptotics of M_T and M_L hold for a *general* but sufficiently smooth $G(|x|,t)$.

Let us consider a particle with $\mathbf{x}=\mathbf{0}$ and $\mathbf{v}=\mathbf{0}$ at $t=0$. In a range of time scales v will be smaller than v_0. Equation (7) thus reduces to $m\,d\mathbf{v}/dt=B^{1/2}\boldsymbol{\eta}$. Thus, trivially, $\langle E_{\rm kin}\rangle\sim\langle\mathbf{v}^2\rangle=Bt/m^2$, whereas $\langle\mathbf{x}^2\rangle^{1/2}\sim t^{3/2}$, as argued in Refs. [1] and [7] by considering $U(\mathbf{x},t)$ which is δ correlated in time; i.e., $\xi_t=0$, and thus $v_0=\infty$. For any *realistic* $U(\mathbf{x},t)$ having $\xi_t>0$ ($v_0<\infty$), this scaling behavior breaks down at the time scale $t_0=v_0^2m^2/B$ when $\langle v^2\rangle$ reaches v_0^2.

We proceed to discuss first the true asymptotic behavior at $t\gg t_0$ for $d=1$. Then Eq. (7) reduces to $m\,dv/dt=[M_L(v)]^{1/2}\eta(t)$. Let us introduce the function

$$\phi(v)=\int_0^v dv'[B/M_L(v')]^{1/2},$$

and make the change $v\to w=\phi(v)$. The dynamics of $w(t)$ is governed by the simple random-walk equation $m\,dw/dt=B^{1/2}\eta(t)$ implying a Gaussian distribution for $w(t)$,

$$P_w(w(t),t)=[2\pi\langle w^2(t)\rangle]^{-1/2}\exp[-(w^2/2)\langle w^2(t)\rangle],$$

with $\langle w^2(t)\rangle=Bt/m^2$. Then, by noting that $v(w)=\phi^{-1}(w)\sim w$, if $|w|\ll v_0$, while $\sim{\rm sgn}(w)|w|^{2/5}$, if

$|w| \gg v_0$, one directly obtains

$$\langle E_{\rm kin}\rangle \sim \langle v^2\rangle \sim \begin{cases} t, & \text{if } t \ll t_0, \\ t^{2/5}, & \text{if } t \gg t_0 \quad (d=1). \end{cases}$$

The mean-square displacement $\langle x^2(t)\rangle = \int_0^t dt_1 \int_0^t dt_2 \times \langle v(t_1)v(t_2)\rangle$ can be calculated by means of the standard random-walk pair distribution function

$$P(w(t_1),w(t_2)) = P_w(w(t_1))P_w(w(t_2)-w(t_1),t_2-t_1)$$

for $t_2 > t_1$. One thus obtains

$$\langle x^2\rangle^{1/2} \sim \begin{cases} t^{3/2}, & \text{if } t \ll t_0, \\ t^{6/5}, & \text{if } t \gg t_0 \quad (d=1). \end{cases}$$

We now discuss $d>1$ by means of Eq. (8). For a particle with $\mathbf{v}=\mathbf{0}$ at $t=0$, $P(\mathbf{v},t)$ will be radially symmetric, i.e., a function of $v=|\mathbf{v}|$ only. Equation (8) implies that the distribution of the velocity magnitude, $P_v(v,t) \sim P(v,t)v^{d-1}$, satisfies

$$\frac{\partial P_v}{\partial t} = \frac{1}{m^2}\frac{\partial}{\partial v}[M_L(v)]^{1/2}\left(\frac{\partial}{\partial v}[M_L(v)]^{1/2} - \frac{d-1}{v}[M_T(v)]^{1/2}\right)P_v . \tag{9}$$

Once again we change v into the variable $w=\phi(v)$, which distribution

$$P_w(w,t) = P_v(v,t)\,dv/dw = P_v[M_L(v)/B]^{1/2}$$

satisfies, by (9),

$$\frac{\partial P_w}{\partial t} = \frac{B}{m^2}\frac{\partial}{\partial w}\left(\frac{\partial}{\partial w} - (d-1)g(w)\right)P_w , \tag{10}$$

with $g(w) = \{[M_T(v)/B]^{1/2}/v\}_{v=\phi^{-1}(w)}$. $g \sim w^{-1}$, if $w \ll v_0$, while $g \sim w^{-3/5}$, if $w \gg v_0$. Equation (10) is equivalent to a standard Langevin process,

$$\frac{dw}{dt} = (d-1)\frac{B}{m^2}g(w) + \frac{B^{1/2}}{m}\eta(t) , \tag{11}$$

where $\eta(t)$ is white noise with $\langle \eta(t)\eta(t')\rangle = \delta(t-t')$. The first term in (11), vanishing in $d=1$, is a drift-type term driving w, i.e., v to infinity as $t \to \infty$. In the absence of the noise, the drift term drives w as $dw/dt \sim w^{-3/5}$, i.e., as $w \sim t^{5/8}$, for $t \gg t_0$. The inclusion of the noise term does not alter this scaling—it only produces a small variation of w of the order $\delta w \sim t^{1/2} \ll \langle w\rangle \sim t^{5/8}$. Thus, the distribution $P_w(w,t)$ is, for $t \gg t_0$, sharply peaked around $w \sim t^{5/8}$. As $v \sim w^{2/5}$, it follows that $P_v(v,t)$ is sharply peaked around $v \sim t^{1/4}$ so $\langle v^2\rangle \sim t^{1/2}$. To summarize our results,

$$\langle E_{\rm kin}\rangle \sim \langle v^2\rangle \sim \begin{cases} t, & \text{if } t \ll t_0, \\ t^{1/2}, & \text{if } t \gg t_0 \quad (d>1). \end{cases}$$

The mean-square displacement, $\langle \mathbf{x}^2(t)\rangle = \int_0^t dt_1 \int_0^t dt_2 \times \Gamma(t_1,t_2)$, with $\Gamma(t_1,t_2) = \langle \mathbf{v}(t_1)\mathbf{v}(t_2)\rangle$, can be obtained by the following line of physical arguments. Equation (7) implies the following equation for the precession of the velocity unit vector $\mathbf{n} = \mathbf{v}/v$:

$$\frac{d\mathbf{n}}{dt} = \frac{1}{mv}[M_T(v)]^{1/2}[\boldsymbol{\eta} - \mathbf{n}(\mathbf{n}\cdot\boldsymbol{\eta})] . \tag{12}$$

Let us introduce the mean free time τ_m such that $\langle \mathbf{n}(t_1)\mathbf{n}(t_2)\rangle \approx 1$, i.e., $\Gamma(t_1,t_2) \approx \langle v^2(t_1)\rangle$, for $|t_1-t_2| < \tau_m$, while $\langle \mathbf{n}(t_1)\mathbf{n}(t_2)\rangle \approx 0$, i.e., $\Gamma(t_1,t_2) \approx 0$, for $|t_1-t_2| > \tau_m$. Thus $\langle \mathbf{x}^2(t)\rangle = \int_0^t dt' v^2(t')\tau_m(v(t'))$. To estimate $\tau_m(v)$, consider a particle with $\mathbf{v}$ along $\mathbf{n}(t_1)$ at $t=t_1$ and split $\mathbf{n}(t)$ into the components $(n_L,\mathbf{n}_T)$, respectively, parallel and perpendicular to $\mathbf{n}(t_1)$. Thus $n_L=1$ and $\mathbf{n}_T=\mathbf{0}$ at $t=t_1$. Then, by (12), for $|\mathbf{n}_T| \ll 1$, $d\mathbf{n}_T/dt \approx \boldsymbol{\eta}_T(t)[M_T(v)]^{1/2}/mv$. Thus $\langle \mathbf{n}_T^2(t_2)\rangle \sim (t_2-t_1)M_T[v(t_1)]/v^2(t_1)$, or, as $M_T \sim v^{-1}$, $\langle \mathbf{n}_T^2(t_2)\rangle \sim (t_2-t_1)/v^3$. For $t_2-t_1=\tau_m$, $\langle \mathbf{n}_T^2(t_2)\rangle \approx 1$ so $\tau_m(v) \sim v^3$. Thus, with $v \sim t^{1/4}$, one eventually obtains $\langle \mathbf{x}^2(t)\rangle^{1/2} \sim t^{9/8}$ for $t \gg t_0$. To summarize our findings,

$$\langle \mathbf{x}^2\rangle^{1/2} \sim \begin{cases} t^{3/2}, & \text{if } t \ll t_0, \\ t^{9/8}, & \text{if } t \gg t_0 \quad (d>1). \end{cases}$$

We proceed to discuss the quantum-mechanical problem, Eqs. (2) and (3), for initial ψ of a wave-packet form. $\langle \mathbf{x}^2\rangle$ and $\langle E_{\rm kin}\rangle$ are expressible in terms of the averaged (over the disorder) density matrix $\rho_{\rm av}(\mathbf{x}_1,\mathbf{x}_2,t) = \langle \psi(\mathbf{x}_1,t)\psi^*(\mathbf{x}_2,t)\rangle$. To calculate $\rho_{\rm av}$ one has to average the product of two quantum-mechanical Green's functions which can be represented by means of a *double* path integral over two sets of paths $\{\mathbf{x}_1(t)\}$ and $\{\mathbf{x}_2(t)\}$ [10]. One can easily show that the average of this path integral over the disorder is a path integral dominated by "paired" paths with $\mathbf{x}_1(t) = \mathbf{x}_2(t)$ [11]. A deeper insight into this pairing is gained by changing variables of the double path integral:

$$\mathbf{x}_1(t) = \mathbf{x}(t) + \tfrac{1}{2}\hbar\mathbf{Y}(t), \quad \mathbf{x}_2(t) = \mathbf{x}(t) - \tfrac{1}{2}\hbar\mathbf{Y}(t) .$$

Thus, the disorder favors paths with $\mathbf{Y}(t)=0$. Moreover, the expansion in powers of $\hbar\mathbf{Y}$, to the leading order in $\hbar$, reduces the quantum problem to the classical path integral (4). Higher-order terms, in $\hbar$, of this quasiclassical expansion are likely to be *irrelevant*—the disorder, by pairing paths, enforces that previously derived *classical superdiffusion scaling laws hold also for the quantum problem.* This *does not* contradict previous studies [2,4-6] indicating ordinary diffusive behavior $\langle \mathbf{x}^2\rangle \sim t$ at large t: They consider a single-band tight-binding model (SBTBM), i.e., a *lattice* model rather than the *continuous* space-time Schrödinger Eq. (3) considered here. SBTBM has *bounded* $E_{\rm kin} < E_{\rm max} = 2\hbar^2/ma^2$, with a the lattice constant. This energy cutoff prevents indefinite acceleration, and, consequently, the superdiffusion at large t. This is, however, an artifact of SBTBM: Inclusion of higher-energy bands would allow for an unlim-

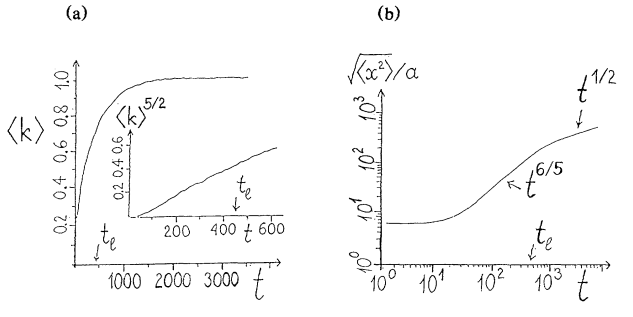

FIG. 1. (a) $\langle k\rangle = 2\langle E_{\rm kin}\rangle/E_{\max}$ vs t (in units of $2\hbar/E_{\max}$). Inset: $\langle k\rangle^{5/2} \sim t$ for $t < t_l \approx 450$. For $t > t_l$, $\langle k\rangle \approx 1$. (b) $\langle x^2\rangle^{1/2}/a \sim t^{6/5}$ for $t < t_l$, and $\sim t^{1/2}$ for $t > t_l$.

ited increase of $E_{\rm kin}$ promoting superdiffusion even for $t \to \infty$ in a *stricto sensu* time-dependent random potential model. We emphasize that the continuous version of such a model is appropriate to the directed wave model [1], or to describe a light test particle wandering in a gas of much heavier particles—in both cases there is no underlying lattice.

To verify the equivalence of the classical and quantum superdiffusive behaviors and to clarify the role of the lattice effects, we simulated 1D SBTBM ((3) with $\nabla^2\psi \to \Delta_l\psi/a^2 = [\psi(x+a) - 2\psi(x) + \psi(x-a)]/a^2$) which can be used to argue about the continuum theory as long as $\psi(x,t)$ is smooth on the scale a. This smoothness can be tested by computing the lattice version of the kinetic energy, $E_{\rm kin} = -(\hbar^2/2ma^2)\sum_x a\psi^*(x)\Delta_l\psi(x)$, satisfying $0 < E_{\rm kin} < E_{\max}$ for a normalized ψ $[\sum_x a|\psi(x)|^2 = 1]$. Thus, $k = 2E_{\rm kin}/E_{\max} < 2$. $k \ll 1$ for ψ smooth on the scale a. $\langle k\rangle = 1$ for a ψ with a *randomly* chosen phase at each lattice site. We solved SBTBM by discretizing t (with step τ) and applying the unitary second-order product formula [12]. We used a smooth initial $\psi(x,0) \sim \exp(-x^2/4x_0^2)$ with $x_0 \gg a$ (Refs. [2,4] used $x_0 \approx a$ so $E_{\rm kin} \approx E_{\max}$ already at $t=0$). We correlated the disorder by imposing that for (x,t) in the range $x_1 < x < x_1 + N_x a$ and $t_1 < t < t_1 + N_t\tau$, $U(x,t)$ is a constant drawn randomly from the interval $|U| < U_0$. The lattice had 3000 sites. The disorder average is obtained by averaging over N samples. Results for $N_x = 3$, $N_t = 5$, $x_0 = 6a$, $\tau = 2\hbar/E_{\max}$, $U_0/E_{\max} = 0.015$, and $N = 6$ samples [13] are shown in Fig. 1. In agreement with our result, $\langle k\rangle \sim \langle E_{\rm kin}\rangle \sim t^{2/5}$ [see the inset of Fig. 1(a)] up to $t = t_l \approx 450$ (in units of $2\hbar/E_{\max}$). For $t > t_l$, $\langle k\rangle$ saturates to 1 ($E_{\rm kin} \approx E_{\max}/2$) indicating complete phase disorder at the scale a, and, thus, the dominance of the lattice effects of SBTBM. Consistently with this, for $t < t_l$, $\langle x^2\rangle^{1/2} \sim t^{1.2\pm0.05} \approx t^{6/5}$, as in our continuous theory, while for $t > t_l$, ordinary diffusion, $\langle x^2\rangle^{1/2} \sim t^{1/2}$, prevails [Fig. 1(b)]. This diffusion is thus *a lattice* effect of SBTBM—inclusion of higher-energy bands would remove the upper energy cutoff and allow for superdiffusive behavior even for $t \to \infty$.

We thank M. Kardar, Y.-C. Zhang, A. Yodh, and D. Sornette for discussions. This work is supported by DARPA under Grant No. Army DAAL 03-89-K-0144.

(a)On leave from Boris Kidrič Institute of Nuclear Sciences, Institute of Theoretical Physics, Vinča, Beograd, Yugoslavia.

[1] S. Feng, L. Golubović, and Y.-C. Zhang, Phys. Rev. Lett. **65**, 1028 (1990); **66**, 2177 (1991).
[2] E. Medina, M. Kardar, and H. Spohn, Phys. Rev. Lett. **66**, 2176 (1991).
[3] J. P. Bouchaud, Europhys. Lett. **11**, 505 (1990).
[4] J. P. Bouchaud, D. Touati, and D. Sornette (to be published).
[5] A. A. Ovchinikov and N. S. Erikhman, Zh. Eksp. Teor. Fiz. **67**, 1474 (1975) [Sov. Phys. JETP **40**, 733 (1975)].
[6] A. Mudhukar and W. Post, Phys. Rev. Lett. **39**, 1424 (1977); S. M. Girvin and G. D. Mahan, Phys. Rev. B **20**, 4896 (1979); Y. Inaba, J. Phys. Soc. Jpn. **50**, 2473 (1981).
[7] A. M. Jayannavar and N. Kumar, Phys. Rev. Lett. **48**, 553 (1982).
[8] M. Kardar and Y.-C. Zhang, Phys. Rev. Lett. **58**, 2087 (1987).
[9] P. C. Martin, E. D. Siggia, and H. A. Rose, Phys. Rev. A **8**, 423 (1973); C. De Dominicis, Phys. Rev. B **18**, 4913 (1978).
[10] R. P. Feynman and A. R. Hibs, *Quantum Mechanics and Path Integrals* (Wiley, New York, 1965).
[11] Similar path pairing occurs also in so-called complex directed polymers, E. Medina, M. Kardar, Y. Shapir, and X.-R. Wang, Phys. Rev. Lett. **62**, 941 (1989); Y.-C. Zhang, *ibid.* **62**, 979 (1989).
[12] H. De Raedt, Comput. Phys. Rep. **7**, 1 (1987).
[13] Our data indicatê a significant self-averaging: $\langle E_{\rm kin}\rangle$ and $\langle x^2\rangle$ are nearly the same for $N=1$ as for $N=6$ samples. Consistent with this is a rather insignificant wandering of the wave-packet *center* (see Ref. [1]) which we find to be subdominant to the wave-packet spreading, $\langle x^2\rangle^{1/2}$.