Multiscaling in Multifractals

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Multiscaling is shown to be a consequence of multifractality when a lower cutoff ϵ is introduced in calculations of correlation functions. After a suitable rescaling, the correlation function data for different values of ϵ seem to fall onto a single curve. In the multiscaling regime, however, we show that there is not a unique functional form at varying ϵ, but a spread very close to a single curve. For each ϵ, this curve can be computed analytically in terms of the $f(\alpha)$ spectrum which characterizes the multifractal. Part of this spectrum can thus be obtained by computing only one moment of the weights at $\epsilon \neq 0$.

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The multifractal formalism [1,2], which originated in the context of turbulence and chaotic systems [3], has become a standard tool to analyze phenomena observed in fractal aggregates, semiconductors, disordered systems, and so on [4]. A multifractal object is characterized by a continuous spectrum of indices, $f(\alpha)$, which describe the global scaling structure. To determine this spectrum one usually computes the scaling behavior of generalized correlation functions,

$$C_q(l) \sim l^{\tau(q)} \text{ for } l \to 0\,, \tag{1}$$

as a function of the length scale l. Then the spectrum follows from the Legendre transformation $\tau(q) = \min_\alpha[\alpha q - f(\alpha)]$. In physical systems, the scaling relation (1) holds only down to a characteristic cutoff in the length scale, after which $\ln C_q(l)$ vs $\ln l$ is no longer linear. The cutoff scale can be varied by an external parameter such as the Reynolds number in turbulence. In studies of power spectra of temperature signals in convective turbulence [5] and of distribution functions for avalanches in sandpile models [6], Kadanoff and co-workers have proposed the existence of a single functional form onto which various curves can be mapped, by a so-called multiscaling [7,8] transformation. One then finds a continuum of scaling exponents by plotting $\ln[C_q(l)/C_0]/\ln(R/R_0)$ vs $\ln(l/l_0)/\ln(R/R_0)$, where R is the control system parameter and C_0, R_0, and l_0 are fitting constants.

Our objective is to justify this procedure, by generalizing the arguments of Frisch and Vergassola [9], who showed that the energy spectrum of turbulent flows exhibits multiscaling in an intermediate dissipative regime as a consequence of the existence of a spectrum of viscous cutoffs [4] in the multifractal model for turbulence.

In this Letter, we extend their idea to a general formalism valid for all kinds of multifractals by including the effect of a varying cutoff on the evaluation of correlation functions C_q. A cutoff ϵ is imposed on the weights of the elements (say, e.g., boxes Λ_i of size l) of a partition of a multifractal set. In practice, we disregard a box Λ_i in the sum $C_q(l) = \sum_{\{\Lambda_i\}} p_i(l)^q$—the correlation function of order q—if its probability $p_i(l)$ is less than ϵ. At varying ϵ one thus computes a series of curves, $C_q(l,\epsilon)$. By the multiscaling procedure, they fall onto a spread of curves which are very close to a single scaling curve for small ϵ. The main point of this Letter is to show that it is possible to compute these curves analytically in terms of the $f(\alpha)$ spectrum. This allows us to reconstruct a part of $f(\alpha)$ without calculating the dimension function $\tau(q)$; it is sufficient to compute C_q for only one particular moment q at finite values of ϵ.

The standard multifractal approach can be summarized as follows. In self-similar sets one defines the scaling indices α_i of the probability measure $p_i(l) \sim l^{\alpha_i}$ of a box Λ_i. By the multifractal ansatz [1,2], the number of boxes with index $\alpha_i \in [\alpha, \alpha+d\alpha]$ is $l^{-f(\alpha)}\rho(\alpha)d\alpha$, and the sum for the generalized correlation function C_q can be written as

$$\begin{aligned} C_q(l) &= \int_{\alpha_{\min}}^{\alpha_{\max}} l^{\alpha q - f(\alpha)} \rho(\alpha)\, d\alpha \\ &\sim l^{\tau(q)} \text{ for } l \to 0\,, \end{aligned} \tag{2a}$$

where $\rho(\alpha)$ is a smooth function independent of l, and $\tau(q) = q\alpha_{\rm SP} - f(\alpha_{\rm SP})$ is given by the saddle-point estimate, i.e., by the value $\alpha_{\rm SP}$ at which the function $\alpha q - f(\alpha)$ reaches its minimum. Therefore, $\alpha_{\rm SP}$ is obtained by inverting the relation $q(\alpha) = f'(\alpha)$. We observe that each moment picks up a particular index value since one has $\alpha_{\rm SP}(q) = \tau'(q)$. The higher the moment q, the smaller the index α, i.e., $q(\alpha)$ is a nonincreasing function of q and $f(\alpha)$ is convex.

However, in the presence of a cutoff ϵ, an empty box cannot be distinguished from a box with probability less than ϵ; a cutoff thus makes it impossible to determine the large α values corresponding to the less probable regions. If $p_i < \epsilon \equiv l^{\bar{\alpha}}$, then the boxes with indices $\alpha > \bar{\alpha} \equiv \ln\epsilon/\ln l$ are not present in the sum for C_q, since they are assumed to be empty, and the upper integration limit in (2a) becomes

$$C_q(l,\epsilon) = \int_{\alpha_{\min}}^{\bar{\alpha}} l^{\alpha q - f(\alpha)} \rho(\alpha)\, d\alpha\,. \tag{2b}$$

This does not modify the saddle-point estimate if $\alpha_{\rm SP}$

$=\tau'(q)$ is still inside the integration interval. However, when $\alpha_{SP}(q)$ is larger than $\bar{\alpha}$, the minimum of the function $\alpha q-f(\alpha)$ in the integration interval is reached at the upper limit $\bar{\alpha}$. We thus obtain

$$C_q(l,\epsilon)\sim\begin{cases} l^{\tau(q)} \text{ if } \bar{\alpha}>\alpha_{SP}\,, \\ l^{q\bar{\alpha}(\epsilon,l)-f(\bar{\alpha}(\epsilon,l))} \text{ if } \bar{\alpha}\leq\alpha_{SP}\,. \end{cases} \tag{3}$$

The *multiscaling* for $\bar{\alpha}\leq\alpha_{SP}$ is a pseudoalgebraic law [9], since it exhibits a power law with a slowly varying exponent, proportional to the logarithm of the length l.

We define the rescaling procedure by considering $F_q\equiv\ln C_q/\ln\epsilon$ as a function of $\theta=\ln l/\ln\epsilon=1/\bar{\alpha}$,

$$F_q(\theta)=\begin{cases} \theta\tau(q) \text{ if } \theta<1/\alpha_{SP}\,, \\ q-\theta f(1/\theta) \text{ if } \theta\geq 1/\alpha_{SP}\,. \end{cases} \tag{4}$$

The first regime is the *single*-scaling regime where the function $F_q(\theta)$ is a straight line with slope $\tau(q)$. In the second regime F_q bends over to the multiscaling region, where the correlation functions for varying values of ϵ are determined by the $f(\alpha)$ spectrum.

The above argument uses only the saddle-point approximation and in the next step we also take the leading corrections into account. These corrections cannot be neglected in the multiscaling regime because when $\bar{\alpha}<\alpha_{SP}$ the first derivative does *not* vanish in the expansion of the exponent in (2) around $\bar{\alpha}$. For $\alpha<\alpha_{SP}$ one therefore has

$$C_q=l^{q\bar{\alpha}-f(\bar{\alpha})}I_q(\alpha_{\min},\bar{\alpha})\,,$$

where, to first order in $\alpha-\bar{\alpha}$,

$$I_q(\alpha_{\min},\bar{\alpha})=\int_{\alpha_{\min}}^{\bar{\alpha}} l^{[q-f'(\bar{\alpha})][\alpha-\bar{\alpha}]}\rho(\alpha)d\alpha\,. \tag{5}$$

Note that the first derivative $d[\alpha q-f(\alpha)]/d\alpha=q-f'(\alpha)<0$ for $\alpha<\alpha_{SP}(q)$ and vanishes for $\alpha=\alpha_{SP}(q)$. Using the relation $l=\epsilon^{1/\bar{\alpha}}$ we finally obtain

$$I_q(\alpha_{\min},\bar{\alpha})=\frac{1-\epsilon^{[f'(\bar{\alpha})-q](\bar{\alpha}-\alpha_{\min})/\bar{\alpha}}}{|\ln\epsilon|[f'(\bar{\alpha})-q]/\bar{\alpha}}R(\bar{\alpha})\,. \tag{6}$$

In the limit $\bar{\alpha}\to\alpha_{\min}$, $I_q(\alpha_{\min},\bar{\alpha})$ must approach unity, because when only boxes with $\alpha=\alpha_{\min}$ have survived the cutoff procedure, one has a homogeneous fractal where $C_q\sim l^{\alpha_{\min}q-f(\alpha_{\min})}$. In this limit the saddle-point calculation [3] becomes exact. This requirement implies that the most singular part of $R(\bar{\alpha})$ is $1/(\bar{\alpha}-\alpha_{\min})$, so we assume $R(\bar{\alpha})=1/(\bar{\alpha}-\alpha_{\min})$ in (6), which is sensible at least for $\bar{\alpha}$ not too far from $\alpha_{\min}$. Therefore, for $\theta\geq 1/\bar{\alpha}$, there is an additive correction to (4), so that

$$F_q(\theta)=q-\theta f(1/\theta)+\frac{\ln I_q(\alpha_{\min},1/\theta)}{\ln\epsilon}\,. \tag{7}$$

To illustrate how the formalism works in practice, we consider examples of two-scale Cantor sets where $f(\alpha)$ is known exactly [2]. We generate sets where, at every stage of the construction, an interval is split into two intervals, each of which covers a fraction β of the previous interval. One assign a probability p_1 to one of the intervals and $p_2=1-p_1$ to the other. At the nth stage of the construction, the correlation sum takes the form ($l\equiv\beta^n$)

$$C_q(l)=\sum_{m=1}^{n}\binom{n}{m}p_1^{mq}p_2^{(n-m)q}\,. \tag{8}$$

With the cutoff ϵ, $C_q(l,\epsilon)$ is obtained by omitting terms in (8) whose probabilities $p_1^m p_2^{n-m}$ are less than ϵ. Figure 1 shows plots of $\ln C_2(l,\epsilon)$ vs $\ln l$ for various values of ϵ. The deviations from the linear scaling take place in a length scale range which increases with the cutoff. Figure 2 shows the rescaled data, plotting $\ln C_q/\ln\epsilon$ vs $\ln l/\ln\epsilon$ for $q=2$ [Fig. 2(a)] and $q=-2$ [Fig. 2(b)]. We get a very good agreement between the data and the theoretical prediction, Eq. (7). Our estimate of $R(\bar{\alpha})$ fails only for $\bar{\alpha}$ close to α_{SP}. In other words, we expect that the function $F_q(\theta)$ turns away from linear behavior at θ^* given by

$$1/\theta^*=\alpha_{SP}-\ln[R(\alpha_{SP})(\alpha_{SP}-\alpha_{\min})]/|\ln\epsilon|\,,$$

and not at $\theta^*=1/\alpha_{SP}$. Again, it is important to note that the multiscaling curve (4) without the cutoff-dependent corrections (7) is only a reasonable approximation. Nevertheless, the leading corrections (7) are so slightly dependent on ϵ that they are well approximated by a single curve in the range of cutoffs considered; see Figs. 2 and 3.

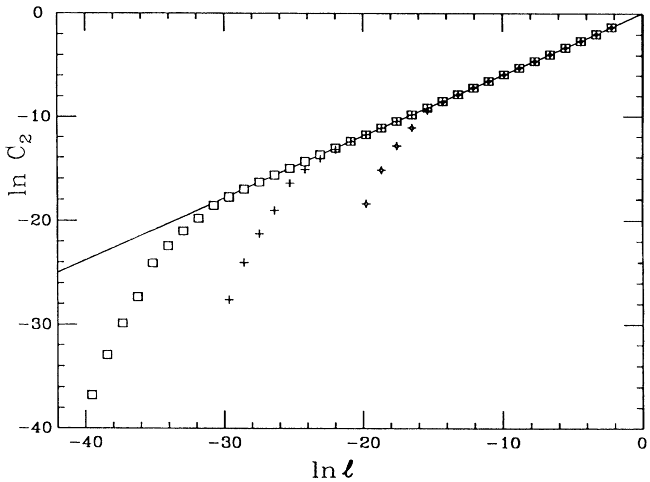

FIG. 1. $\ln C_2$ vs $\ln l$ for the two-scale Cantor set with $\beta=\frac{1}{3}$ and $p_1=0.6$. The fractal dimension of the set is $D_F=-\tau(0)=\ln 2/\ln 3$, and $\alpha_{\min}=\ln p_1/\ln\beta$. The straight line has slope $\tau(2)=0.595\ldots$ The data are obtained for $\epsilon=10^{-4}$ (crossed circles), $\epsilon=10^{-6}$ (crosses), and $\epsilon=10^{-8}$ (squares).

As an additional, less trivial test of the formalism, we have applied it to the accumulation point of period-doubling bifurcations [10]. Consider the set of points generated by the dynamical system $x_{n+1}=\lambda(1-2x_n^2)$ at $\lambda=0.837\,005\,134\ldots$ The corresponding $f(\alpha)$ spectrum

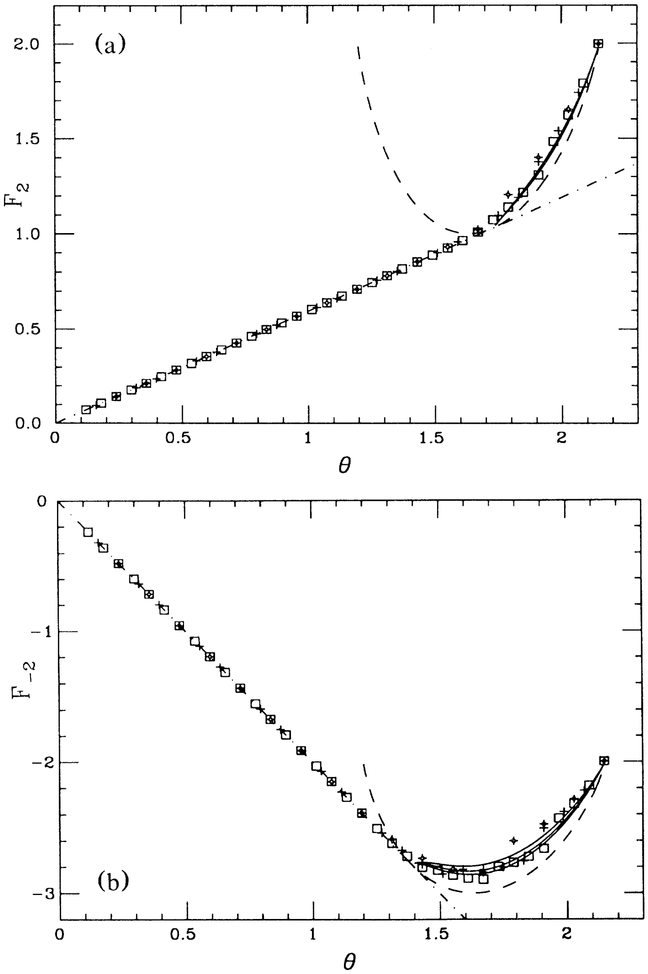

FIG. 2. (a) Multiscaling transformation for the data of Fig. 1: $F_q(\theta)=\ln C_q/\ln\epsilon$ vs $\theta=\ln l/\ln\epsilon$, with $q=2$. The straight dash-dotted line is $\tau(q)\theta$, with $\tau(2)=0.595\ldots$, $\alpha_{\rm SP}=\tau'(2)=0.578\ldots$. The multiscaling regime is expected in the interval $\theta^*=1/\alpha_{\rm SP}=1.728\ldots$ up to $\theta_{\rm max}=1/\alpha_{\rm min}=2.150\ldots$. The dashed parabolic line is the curve $q-\theta f(1/\theta)$ using the explicit form of the $f(\alpha)$ spectrum for the two-scale Cantor set. The solid lines are the leading correction (7), corresponding to $\epsilon=10^{-4}$, 10^{-6}, and 10^{-8}. They are practically indistinguishable. (b) For the same Cantor set, $F_q(\theta)=\ln C_q/\ln\epsilon$ vs $\theta=\ln l/\ln\epsilon$, with $q=-2$. The data are obtained for $\epsilon=10^{-4}$ (crossed circles), $\epsilon=10^{-6}$ (crosses), and $\epsilon=10^{-8}$ (squares). The lines correspond to those of 2(a) where now $\tau(-2)=-2.002\ldots$, $\alpha_{\rm SP}=\tau'(-2)=0.7204\ldots$, and $\theta^*=1/\alpha_{\rm SP}=1.387\ldots$.

is known to high precision [2]. We generate 10^6 points, partition the set into boxes of size l, and estimate $C_2(l,\epsilon)$ for various values of ϵ. Figure 3 shows $\ln C_2/\ln\epsilon$ vs $\ln(l/l_0)/\ln\epsilon$, where l_0 is a parameter such that $C_2(l)=(l/l_0)^{\tau(2)}$ in the linear part of the scaling. We have used ϵ values which are rather large with respect to the case of Fig. 2. However, the agreement with the theoretical prediction (7) is quite good.

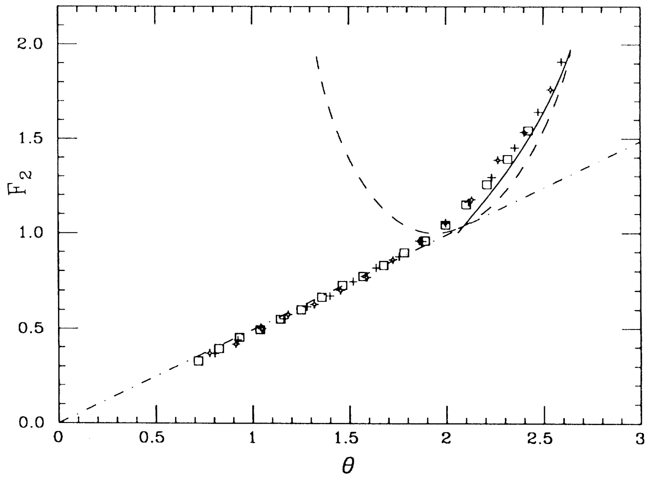

FIG. 3. $F_2(\theta)=\ln C_2/\ln\epsilon$ vs $\theta=\ln(l/l_0)/\ln\epsilon$ for the period-doubling repeller. The data for $C_2(l,\epsilon)$ are obtained at $\epsilon=0.0015$ (squares), $\epsilon=0.003$ (crosses), and $\epsilon=0.006$ (crossed circles). The lines are the same as in Fig. 2, with $\tau(2)=0.495$.

In dynamical systems there is only one adjustable parameter, i.e., l_0 which in the two-scale Cantor set is equal to unity by construction. In general physical phenomena, one, of course, has to consider fitting parameters [5,6,8] C_0, ϵ_0, l_0, defining the multiscaling as $\ln[C_q(l)/C_0]/\ln(\epsilon/\epsilon_0)$ vs $\ln(l/l_0)/\ln(\epsilon/\epsilon_0)$, but for normalized probabilities $C_0=1$. The formalism is at present being applied to the energy spectrum of turbulence [9] in a shell model [11]. The initial results are encouraging and will, together with details on the period-doubling investigation and other results, appear in a forthcoming publication.

In conclusion, we have related the multiscaling observed in multifractals to variations in a physical cutoff parameter. Our main results are condensed in Eqs. (4) and (7) which indicate that the correlations for various values of the cutoff ϵ are well approximated (although in no limit exactly) by a scaling curve, determined by a portion of the $f(\alpha)$ spectrum. Finally, it seems to us an open and very promising problem whether one can determine the underlying multifractal structure (if any) when multiscaling is empirically observed in a physical system, as in Refs. [5], [6], and [12]. However, it is not clear whether multiscaling must be always related to multifractality or is a much more general phenomenon.

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