Composite Edges in the $\nu=\frac{2}{3}$ Fractional Quantum Hall Effect

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The quantum Hall effect is due to discontinuities in the chemical potential at certain bulk electron densities. It has been conjectured that the resulting edge states in some fractional Hall gaps can have an intricate composite structure which reflects the correlations in the incompressible bulk fluid. We present direct numerical evidence obtained from exact-diagonalization studies in favor of this conjecture. We show that the states on the $\nu=\frac{2}{3}$ fractional quantum Hall edge constitute a two-component system in which a $\nu=\frac{1}{3}$ droplet of holes is contained within a $\nu=1$ droplet of electrons.

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The allowed kinetic energies for electrons moving in two dimensions in a perpendicular magnetic field are quantized into Landau levels separated by $\hbar\omega_c$, where $\omega_c=eB/mc$ is the cyclotron frequency. The quantum Hall effect [1,2] occurs in these systems whenever the chemical potential has a discontinuity (so that the system is incompressible) at an electron density n which depends on magnetic field. A system of noninteracting electrons is incompressible when the Landau-level filling factor $\nu=n2\pi l^2$ is an integer and the chemical potential jumps from one allowed kinetic energy to another. (Here $l^2=\hbar c/eB$.) The quantum Hall effect also occurs at a series of fractional filling factors [3]; the incompressibilities in this case are due to interactions among electrons in the same Landau level. It is possible [4,5] to form many-body states in the lowest Landau level in which no pair of electrons is ever found in a state of relative angular momentum (RAM) $m-2$ only if $\nu\leq 1/m$. This property is responsible [6] for the incompressibility and hence the fractional quantum Hall effect (FQHE) at $\nu=1/m$. The incompressibility at $\nu=1-1/m$ then follows from particle-hole symmetry (see below). The fact that these incompressibilities occur at magnetic-field-dependent densities requires the existence, in an open system, of edge states which evolve with magnetic field in such a way as to provide the required change in density. It is this requirement which leads to the fractional quantum Hall effect [2,7] and which is the basis of Chern-Simons effective theories of edge excitations and the classification of incompressible Hall states in terms of Kac-Moody algebras [8–10]. In this Letter we provide numerical evidence in support of suggestions [8,11] that the edge is composite, that is, that the Hilbert space relevant to fractional edge excitations is the product of (in general, more than one) Hilbert spaces for chiral Luttinger liquids. (Integer edges can be considered [12–15] as the product of n Luttinger liquids for $\nu=n$.)

Let us consider the cases $\nu=\frac{1}{3}$ and $\frac{2}{3}$. The calculations which we describe below are made using the hard-core model in which a pair of electrons interact only when they are in a state of RAM 1. (In the language of Haldane's pseudopotentials [16] only $V_1\neq 0$.) In the bulk the chemical potential for this model jumps from 0 to $\sim V_1$ when ν crosses $\frac{1}{3}$ and pairs of electrons are forced to occupy RAM 1. The $\nu=\frac{1}{3}$ FQHE and its daughters occur, for $\frac{1}{3}\leq\nu\leq\frac{2}{3}$, as long as the true interaction is sufficiently close [16] to this ideal model for the chemical potential gaps to survive [6]. In this model, any many-body state in the gap must avoid RAM 1 and must therefore be expressible in the form [11,17–19]

$$\Psi_\alpha[z]=Q_\alpha[z]\prod_{i<j}(z_i-z_j)^m\prod_k\exp(-|z_k|^2/4)\,, \quad (1)$$

where $m=3$ and $Q_\alpha[z]$ is a symmetric polynomial of homogeneous degree M. For edge excitations, M is small compared to N, the number of electrons in the system. [For $M\ll N$ and $M=1,2,\ldots,7$, the number of independent excitations is 1, 2, 3, 5, 7, 11, and 15, respectively. For a single quasihole excitation $M\sim N$, while for a change of filling factor $M\sim N^2$. We choose to work in the symmetric gauge where the single-particle eigenstates are labeled by angular momentum $m=0,1,2,\ldots$, $\phi_m(z)=(2\pi m!2^m)^{-1/2}z^m\exp(-|z|^2/4)$, and $z=(x+iy)/l$ is the electron coordinate.]

A rather complete description of the $\nu=1/m$ edge is provided by a mapping to the integer edge. The $M=0$ state is the Laughlin wave function [4], the most compact state which (for $m=3$) avoids RAM 1. The low-lying excitations for $\nu=1/m$ can be mapped one to one onto the low-lying edge states for $\nu=1$, which are the particle-hole excitations of a filled Landau level. In each case, the excitations with total angular momentum M can be put into one-to-one correspondence with the set of symmetric polynomials of homogeneous degree M [14,15,17,18], and so form one branch.

For $\nu=\frac{2}{3}$ the number and nature of edge excitations does not follow as readily from the microscopic origin of the bulk gap, which in this case is understood in terms of particle-hole symmetry. A $\nu=\frac{1}{3}$ state has an edge where the electron density falls from $\nu=\frac{1}{3}$ to zero. Imagine constructing a $\frac{2}{3}$ state by particle-hole conjugation of the $\frac{1}{3}$ state. The resulting electronic density then goes from $\nu=\frac{2}{3}$ to 1 at the same place, and so there must be a second, outer edge at which the density falls from $\nu=1$ to zero. It has been conjectured [11,17] that for a physically smooth confining potential the outer edge can also support excitations so that the relevant many-body wave functions are given by

$$\Psi_{\beta,\alpha}[z] = Q_\beta[z] C(\Psi_\alpha[z]) , \quad (2)$$

where $C(\Psi[z])$ denotes particle-hole conjugation with reference to the truncated Hilbert space (see below). In the present work we provide the first direct evidence that this picture of two separated edges, supporting two branches of excitations, is in fact correct. We find the exact ground state and low-lying excited states for small $\nu = \frac{2}{3}$ systems of electrons confined by a potential in a disk geometry.

We begin with an examination of particle-hole conjugation for a finite system. Consider the Hamiltonian H for a system of electrons on a disk, lying in the lowest Landau level, subject to the hard-core (V_1) interaction and some confining potential $W(|z|)$. Replacing the electron creation operator $e^\dagger$ by a hole annihilation operator h, the Hamiltonian in the hole representation is

$$H = H_{hh} - \sum_{m=0}^{M_o} (2D_m + W_m) h_m^\dagger h_m + \sum_{m=0}^{M_o} (W_m + D_m) , \quad (3)$$

where H_{hh} is the hole-hole interaction and

$$2D_m = \sum_{m'=0}^{M_o} (\langle mm'|V|mm'\rangle - \langle mm'|V|m'm\rangle) . \quad (4)$$

Here the external potential $W(|z|)$ has matrix elements $\langle m'|W|m\rangle = W_m \delta_{mm'}$ in the symmetric gauge. Since the expected radius of the mth single-particle state increases with m, a confining potential $W(|z|)$ will lead to coefficients W_m which also increase with m. In order to have a finite number of holes, the Hilbert space must be truncated at $m = M_o$, which is equivalent to introducing an unphysical hard edge at M_o. Thus suitable W_m's must be chosen which keep the electrons away from M_o (see below). Above, $-2D_m$ represents the attractive interactions between holes and the electrons present in the hole vacuum. This potential slowly declines in magnitude as the hard edge is approached and tends to confine holes to the region occupied by electrons. (For the hard-core model and large M_o, $D_m \to 2$ for small m and $D_m \to 1$ as $m \to M_o$.) H_{hh} has exactly the same form as the electron-electron interaction (i.e., the holes interact via the same hard-core or V_1 interaction), so the bulk gap for $\nu = \frac{2}{3}$ is a result of the holes arranging themselves to avoid RAM 1.

For the choice of confining potential $W_m = -D_m$, H is completely particle-hole symmetric. It follows that in this case the edge states are simply the particle-hole conjugates of the electron edge states discussed above, and so there is only one branch.

We now consider the physically relevant case of smooth edge potentials. If the electron density has already been pushed to zero by W_m by the time M_o is reached, then the truncation plays no essential role. As long as it satisfies this smoothness condition, the detailed form of W_m does not matter greatly. Here we will use a simple edge described by

$$W_m = \begin{cases} 0, & \text{if } m \le M_e, \\ (m - M_e)S, & \text{if } m > M_e . \end{cases} \quad (5)$$

The results reported here are for $M_o = 18$, $M_e = 12$, and $S = 0.35V_1$. There are nineteen single-particle states in the Hilbert space. Note that $M_T = \sum_m m c_m^\dagger c_m$ is a good quantum number for any $W(|z|)$ and $V(|z_1 - z_2|)$.

In Fig. 1 we show the single-particle occupation numbers $\langle n_m\rangle$ for the exact ground states with from two to six holes on the nineteen-state disk. In each case we identify a "bulk" $\nu = \frac{2}{3}$ state near the center, followed by composite edges. Added holes go either to a hole droplet at the center or to the edge of the system. Within the hole droplets $\langle n_m\rangle$ is nearly independent of the number of holes at the edge of the system. Within the two- and three-hole droplets $\langle n_m\rangle \sim 1 - \langle n_m\rangle_L$, where $\langle n_m\rangle_L$ is the angular momentum distribution in the $\nu = \frac{1}{3}$ Laughlin state for the same number of particles [20]. (The characteristic bump in $\langle n_m\rangle_L$ near the edge reflects the possibility of avoiding RAM 1 at a locally higher electron density near the edge.)

In the ground state, single-particle states beyond the inner droplet have quite well-defined occupancies (Fig. 1). This is also true for low-lying excitations and permits us to identify two separate excitation branches. In Fig. 2

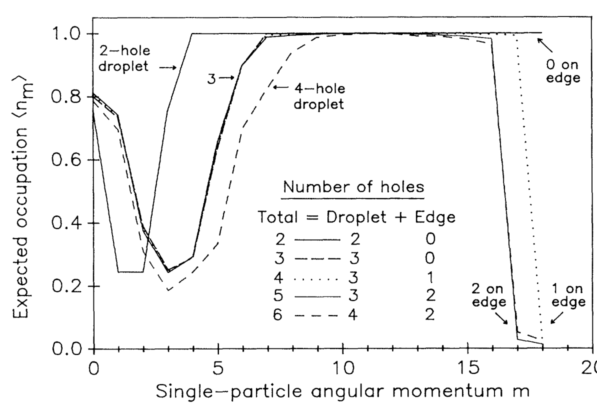

FIG. 1. Occupations $\langle n_m\rangle = \langle e^\dagger e\rangle$ in the ground state for various fillings, with a confining potential $W_m = 0.35(m-12)$ for $m > 12$. Notice that added electrons can go either to an inner or an outer edge. The six-hole ground state, for example, has a four-hole $\nu = \frac{1}{3}$ droplet plus two holes at the outer edge. An added electron goes to the inner edge, so that the five-hole ground state consists of a three-hole droplet plus the same two holes at the outer edge. The inner holes try to form a $\nu = \frac{1}{3}$ droplet; they are very nearly particle-hole conjugates of Laughlin states for the same number of electrons.

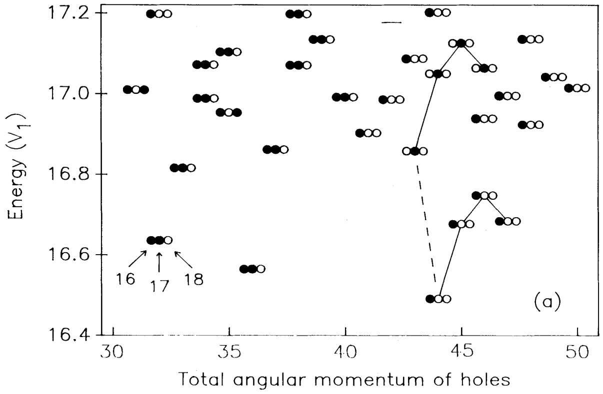

FIG. 2. Excitation spectrum for five holes and fourteen electrons. The three dots associated with each eigenstate show the occupancy of the three outermost single-particle states, at $m=16$, 17, and 18. Nominally empty states (with $\langle n_m \rangle < 0.1$) are labeled with an open circle, while nominally occupied states (with $\langle n_m \rangle > 0.9$) are labeled with closed circles. In certain sequences of excitations, such as those connected by solid lines, the outer edge does not change. Thus the energy changes are a result of excitations in the droplet and at the inner edge. The rightmost two sequences marked here have different outer edges, but have similar three-hole droplets. The sequences are parallel because their droplets are undergoing identical excitations. An excitation at the outer edge is given by the offset between these two sequences (the dashed line). Going up the dashed line, the holes' total angular momentum changes by -1 because one hole moves inward (or one electron moves from $m=16$ to $m=17$). The vertical distance going up the dashed line is the excitation energy required to do this.

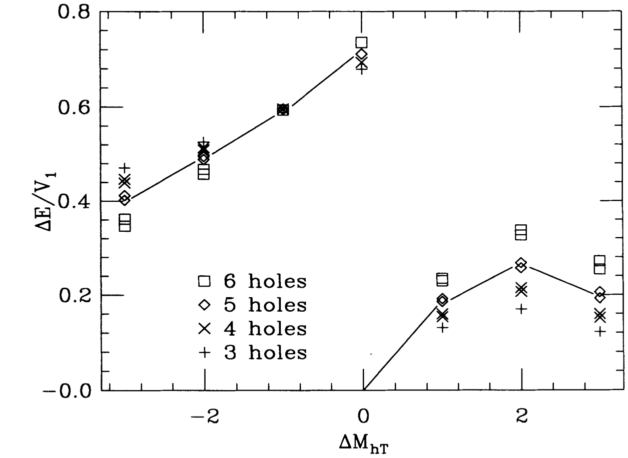

FIG. 3. Excitation energies of the inner-edge branch for three-hole droplets. These are found by comparing two states with the same nominal outer-edge configuration but different states of the inner hole droplet. Calculations with a total of three, four, five, or six holes give nearly the same energies, evidence of the independence of the two edges. The lowest-energy excited state is shown for each value of ΔM_{hT} (the change in total hole angular momentum). The lines are guides to the eye.

the low-energy spectrum is shown as a function of the total hole momentum M_{hT} for a system with five holes and hence fourteen electrons. The labeling indicates the occupancies of the outermost three single-particle states (those with $m=16,17,18$), with open dots indicating holes and solid dots electrons. (An occupancy of less than 0.10 is labeled as a hole and greater than 0.90 as an electron.) The ground state is represented as ●○○, in agreement with Fig. 1. The ground state has $M_{hT}=44$ as expected for a three-hole Laughlin droplet [$M_{hT}=3H_h(N_h-1)/2=9$] plus two outer holes at $m=17,18$. Let us classify a number of the low-lying excitations in Fig. 1. (1) The states connected to the ground state by a solid line (the sequence with $M_{hT}=45,46,47$) are excitations on the inner edge. (Note that there is one low-lying state with an angular momentum increase of 1 and two with an increase of 2, as expected. The lowest state with $M_{hT}=44+3$ is a "bulk" quasihole excitation of the three-hole droplet.) We emphasize that, although some mixing between inner and outer edges does occur in the exact eigenstates, the quantum numbers of the unmixed states are preserved. (2) The lowest-energy state at $M_{hT}=44-1$, connected to the ground state by a dashed line in Fig. 1, is the $\Delta M_T=1$ outer-edge excitation. (Increasing angular momentum for electrons corresponds to decreasing angular momentum for holes.) The sequence of states connected to this one by solid lines is nearly identical, except for a constant shift, to the sequence connected to the ground state. This shows that the excitation energies on one edge are nearly independent of the excitation state on the other edge. In each case, the outer edge is unchanging, so the excitations are taking place in the three-hole droplet. (3) The second lowest-energy state in Fig. 2 occurs at $M_{hT}=36$, which arises from a four-hole Laughlin droplet with $M_{hT}=18$ plus a hole at $m=18$. This represents an excitation in which charge is transferred from the outer edge to the inner edge. The low-energy ●●○ states at $M_{hT}=37$–40 are excitations on the edge of the four-hole Laughlin droplet. Again there is one excitation with $\Delta M_{hT}=1$ and two with $\Delta M_{hT}=2$. (4) The state at $M_{hT}=32$ is a bulk quasiparticle excitation in the four-hole droplet. One can identify all low-lying states in this manner.

Now let us examine the two branches quantitatively. For excitations on the outer edge with $\Delta M_T=1$, ΔE ranges from $0.36V_1$ to $0.41V_1$, compared to the value $0.35V_1$ which would be expected for noninteracting electrons with our confining potential. The velocity of the edge mode, $\Delta E/\Delta M_T$, is positive, and its magnitude is only weakly renormalized by electron-electron interactions. An example of excitation energies for an inner-edge branch is shown in Fig. 3, for three-hole droplets with three to six total holes. The low-energy excitations result from expanding the hole droplet ($\Delta M_{hT}>0$) which means $\Delta M_T<0$ for electrons. Thus the velocity

$\Delta E/\Delta M_T$ is negative for the inner branch. For the three-hole droplets in Fig. 3, the velocity is $\approx -0.18V_1$; for the four- and five-hole droplets, it is $\approx -0.25V_1$. The excitation with $\Delta M_{hT} = -3$ represents a quasielectron excitation which allows compression of the hole droplet. All compressive excitations ($\Delta M_{hT} \leq 0$ in Fig. 3) lie above a gap of order V_1. We can also define a chemical potential for each branch by computing the energy to add an electron to either edge. For example, the lower sequence connected by lines in Fig. 2 can be compared to a similar sequence of six-hole states with the three outer states vacant, to find the energy to add an electron at $m=15$. Proceeding in this manner, we can calculate the energy to add an electron at each m in several ways, with nearly the same result. Adding an electron at m at the outer edge turns out to cost an energy equal to the confining potential plus an interaction energy $\approx 1.7V_1$, nearly independent of m. (The chemical potential jump at $\nu = \frac{2}{3}$ for the hard-core model in the thermodynamic limit is from about $2V_1$ to $4V_1$.) One can, similarly, find the energy to add an electron to the inner edge, shrinking the inner hole droplet. These chemical potentials will be identical in the thermodynamic limit. One can also assign to each branch a charge equal to the inverse change in flux quanta enclosed as an electron is added to the branch [11]. Adding an electron to the outside fully occupies one more $|m\rangle$, so one more quantum of flux is enclosed, and $e^*_{\rm outer} = e$. The inner droplet changes area just like a Laughlin wave function. Adding an electron to the inner edge (i.e., removing one hole from the inner droplet) thus reduces the droplet's area by three flux quanta, corresponding to a charge of $e^*_{\rm inner} = -e/3$. These satisfy the sum rule $\sum_i e_i^*/e = \frac{2}{3}$, as expected [11,21].

We have identified two physically separated branches of edge excitations for a bulk $\frac{2}{3}$ state. These are a natural consequence of the origin of the $\frac{2}{3}$ state, which is obtained by particle-hole conjugation of the $\frac{1}{3}$ state. One branch lies on the outer edge, where $\langle n_m \rangle$ falls from 1 to 0, and corresponds to electrons confined by an external potential. Excitations on this branch have positive velocity and integral charge. The other branch lies on the inner edge, where $\langle n_m \rangle$ rises from $\frac{2}{3}$ to 1, and corresponds to the edge branch for a $\frac{1}{3}$ state of holes. The hole droplet is repelled and confined by the electronic edge. The inner-edge excitations have negative velocity and fractional charge. The number of edge branches obtained here and their general features (velocities, charges, and certain sum rules) support the earlier conjectures of MacDonald [11] and Wen [8].

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[1] K. von Klitzing, G. Dorda, and M. Pepper, Phys. Rev. Lett. **45**, 494 (1980).

[2] For an elementary introduction, see A. M. MacDonald, *A Perspective on the Quantum Hall Effect* (Klewer, Boston, 1989).

[3] D. C. Tsui, H. L. Störmer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).

[4] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).

[5] S. A. Trugman and S. A. Kivelson, Phys. Rev. B **31**, 5280 (1985).

[6] For a recent numerical study in support of this claim, see C. Gros and A. H. MacDonald, Phys. Rev. B **42**, 10811 (1990).

[7] For a more mathematical discussion of this point, see Joseph E. Avron, Ruedi Seiler, and Barry Simon, Phys. Rev. Lett. **65**, 2185 (1990).

[8] X. G. Wen, Phys. Rev. Lett. **64**, 2206 (1990); Phys. Rev. B **41**, 12838 (1990); Phys. Rev. Lett. **66**, 802 (1991).

[9] J. Frölich and A. Zee, Institute for Theoretical Physics, Santa Barbara, Report No. NSF-ITP-91-31 (to be published), and work quoted therein.

[10] N. Read, Phys. Rev. Lett. **65**, 1502 (1990).

[11] A. H. MacDonald, Phys. Rev. Lett. **64**, 220 (1990).

[12] B. I. Halperin, Phys. Rev. B **25**, 2185 (1982).

[13] A. H. MacDonald and P. Středa, Phys. Rev. B **29**, 1616 (1984).

[14] F. D. M. Haldane, Bull. Am. Phys. Soc. **35**, 254 (1990).

[15] Michael Stone, Phys. Rev. B **42**, 8399 (1990).

[16] F. D. M. Haldane, in *The Quantum Hall Effect,* edited by R. E. Prange and S. M. Girvin (Springer-Verlag, New York, 1990), 2nd ed., Chap. 8.

[17] A. H. MacDonald and D. B. Murray, Phys. Rev. B **32**, 2707 (1985).

[18] Dung-Hai Lee and Xiao-Gang Wen, Phys. Rev. Lett. **66**, 1765 (1991).

[19] We remark that for these wave functions the charge transfer associated with neutral particle-hole excitations is $e/3$ while the flux-quantum charge is e. For charged excitations the flux-quantum charge is $e/3$. (Three flux quanta are required to place an additional electron in a given area.) Thus the microscopic wave functions in the gap are qualitatively different from those of the free system of charge-$e/\sqrt{3}$ particles used in the effective theory of Ref. [8] even though both systems have the same Hall conductivity.

[20] The four-hole droplet is squeezed by the potential to a smaller total hole momentum than the Laughlin state, 14 instead of 18.

[21] Wen defines an equivalent sum rule in terms of an alternative "optical" charge [8,19]. The optical charges here are $q_{\rm outer} = e$ and $q_{\rm inner} = -e/\sqrt{3}$. Only the squares of these charges have significance; e.g., the sum rule is $\sum_i {\rm sgn}(v_i) q_i^2/e = \nu$, here yielding $\frac{2}{3}$.