Algebraic Structure of Translation-Invariant Spin- $\frac{1}{2}$ xxz and q-Potts Quantum Chains

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(Received 24 June 1991)

Spin- $\frac{1}{2}$ xxz and q-Potts quantum chains with translation-invariant boundary conditions are analyzed as representations of the periodic Temperley-Lieb-Jones algebra. A connection with the affine Hecke algebra is established and used to find the irreducible content. This analysis provides an explanation for both the degeneracies and the overlap in the spectra of these models.

PACS numbers: 05.50.+q, 02.20.+b

Numerical studies [1] of the finite spin- $\frac{1}{2}$ xxz quantum chain, subject to a class of translation-invariant boundary conditions (hereafter TBC), have revealed a rich pattern of degeneracies in the spectrum of the associated Hamiltonian H. Exact relations between the spectra of the xxz and of the *a*-Potts chains have also been observed [1,2]. Both phenomena are also known to occur for a special choice of non-TBC. For this choice H is given by a sum over the generators of the Temperley-Lieb-Jones (TLJ) algebra. Using the representation theory of the TLJ algebra to analyze these models yields exact information about the degeneracies and the overlap in their spectra [3]. The degeneracies of the xxz spectra can also be computed (for this special BC) from the fact that the quantum group U_q (SU(2)) generates the commutant of the TLJ algebra in this representation. U_q (SU(2)), is, however, not useful for deriving the relations between the xxz and Potts chains.

Exact results about the symmetry properties of the finite xxz and Potts chains with TBC can be found in [2,4]. In [4] it was observed that while U_q (SU(2)) generators do not commute anymore with H, a certain algebraic relation involving H and the generators, from which information about the degeneracies can be obtained, can nevertheless be derived. In [2] a proof of an exact relation between the spectra of the xxz chain and the q-Potts models for one particular choice of TBC and one particular q-Potts charge sector is given. Unfortunately, there seems to be no obvious generalization of this proof to other TBC and/or charge sectors.

Translation-invariant lattice models form an important class of models, especially for making a connection with a continuum field theory description. Yet, as I tried to suggest, our understanding of the finite-lattice symmetries of these systems is lacking, while the results of [1,2,4] give compelling evidence for their existence. Here I would like to view the xxz and q-Potts models as representations of the periodic TLJ algebra (see also [4]). The key idea is to analyze the latter through their relation with the affine Hecke algebra representations. After giving the necessary mathematical background, we will consider the xxz and Potts models. I will only outline the main results and give a few illustrative examples, deferring many details and a more systematic discussion to a longer paper. Periodic TLJ and its relation to affine Hecke.—The TLJ algebra is an associative algebra over \mathbb{C} which is generated by the unit element 1 and e_1, \ldots, e_N . We consider the following relations:

(a)
$$e_i^2 = e_i$$
, (b) $e_i e_{i \pm 1} e_i = \tau e_i$,
(c) $[e_i, e_j] = 0 \forall |i - j| > 1$,
(1)

where $0 < \tau < \infty$ and N is a non-negative integer. The free-ends TLJ algebra, $FTLJ(\tau, N)$ (the "usual" TLJ algebra), is defined by the subset of the relations (1) for which $1 \le i \pm 1 \le N$ on the left-hand side (lhs) of (1b). The periodic TLJ algebra, $PTLJ(\tau, N)$, is defined by the relations (1) with $1 \le i \le N$ and the identifications $0 \equiv N$ and $N+1 \equiv 1$ made on the lhs of (1b) and in computing |i-j| on the right-hand side (rhs) of (1c). While $FTLJ(\tau, N)$ is finite dimensional for finite N, $PTLJ(\tau, N)$ is infinite dimensional if N > 2. Moreover, $FTLJ(\tau, N)$ \subset FTLJ(τ , N+1)—a useful property for the analysis of the irreducible representations (irreps) [5]. There seems to be no simple analog of this property for $PTLJ(\tau, N)$. We now turn to the connection between $PTLJ(\tau, N)$ and the affine Hecke algebra AH(t,N) which offers a way to circumvent this problem since $AH(t,N) \subset AH(t,N+1)$.

A PTLJ (τ, N) algebra automorphism is given by $e_i \rightarrow e_{i+1}$ $(i=1,\ldots,N, N+1=1)$. Assume that a representation of PTLJ (τ, N) is specified together with an invertible operator \vec{C}_1 which realizes this automorphism, namely,

$$\vec{C}_1 e_i \vec{C}_1^{-1} = e_{i+1}, \quad i = 1, \dots, N, N+1 \equiv 1.$$
 (2)

Consider, moreover, the set of N-1 equations

$$\hat{C}_{1}e_{i}\hat{C}_{1}^{-1} = e_{i+1}, \quad i = 1, \dots, N-2,$$

$$\hat{C}_{1}^{2}e_{N-1}\hat{C}_{1}^{-2} = e_{1}.$$
(3)

Obviously, any \vec{C}_1 satisfying (2) is also a solution of (3), but not the other way around. Therefore the existence of \hat{C}_1 is a necessary but not a sufficient condition for the existence of \vec{C}_1 . The set (3) involves only \hat{C}_1 and the generators of FTLJ $(\tau, N-1)$ which is a subalgebra of PTLJ (τ, N) . Defining Hecke algebra generators $g_i = (1+t)e_i - 1$ ($g_i^2 = (t-1)g_i + t$, $g_ig_{i+1}g_i = g_{i+1}g_ig_{i+1}$,

 $[g_i,g_j]=0 \ \forall |i-j|>1)$, where $t \in \mathbb{C}$ $(t\neq 0,-1)$ is a solution of $\tau^{-1}=2+t+t^{-1}$, we find $\overline{C}_1\equiv g_1g_2\cdots g_{N-1} \in \mathrm{FTLJ}(\tau,N-1)$ is a solution of (3). Given any other solution \hat{C}_1 of (3) we define

$$\vec{X} = \vec{C}_1^{-1} \hat{C}_1. \tag{4}$$

Equations (3) are equivalent to the following set of equations for \vec{X} :

$$[X,g_i] = 0 \quad (1 \le i \le N - 2), \tag{5a}$$

$$\vec{X}g_{N-1}\vec{X}g_{N-1} = g_{N-1}\vec{X}g_{N-1}\vec{X}.$$
 (5b)

Equations (5) together with the previous relations for g_1, \ldots, g_{N-1} are nothing but the defining relations for $AH(t^{-1}, N)$ with $\vec{X} = x_N$ [6]. To summarize, all $PTLJ(\tau, N)$ representations for which an invertible \vec{C}_1 exists are also $AH(t^{-1}, N)$ representations with x_N given by (4). Clearly, \vec{C}_1 and $FTLJ(\tau, N-1)$ are sufficient for generating $PTLJ(\tau, N)$. An interesting complication arises in some of the q-Potts representations for which an invertible \vec{C}_1 does not exist. Nevertheless, there exists in these representations an invertible \vec{C}_2 , generating the automorphism $e_i \rightarrow e_{i+2}$, of the form $\vec{C}_2 = (\hat{C}_1 \vec{X})^2$.

AH(t,N) quotients.—We will describe a family of finite-dimensional irreps of AH(t,N) which are relevant for the Potts and xxz chains with TBC. They arise from a finite-dimensional quotient of AH(t,N), denoted $SAH(\tau,b,c,N)$, for which the following holds:

$$x_{N}^{2} = bx_{N} + c,$$

$$e_{N-1}x_{N}e_{N-1} = \frac{b}{1+t}e_{N-1}, \quad b^{2} + 4c \neq 0,$$
(6)



FIG. 1. (a) The Bratteli diagram for the generic SAH (τ, b, c, N) with $N \le 5$. (b) The double-cut Bratteli diagram describing SAH(3,3,N) with $N \le 5$. The representations on the even floors appear in the Potts-3 model with Z_3 preserving TBC.

where b,c are numerical parameters. In [7] a detailed analysis of the structure was carried out for the special case b=0, c=1. It can be easily extended to the more general case at hand. We now summarize the main results. The dependence of the structure on the parameters is determined by the zeros of the following polynomials:

$$P_{m}^{\pm}(\gamma,b,c) = (2\cos\gamma)^{-m} \left(e^{im\gamma} - \frac{2\cos\gamma f^{\pm} - e^{i\gamma}}{2\cos\gamma f^{\pm} - e^{-im\gamma}} e^{-im\gamma} \right), \quad 2\cos\gamma f^{\pm} - e^{-i\gamma} \neq 0,$$

$$f^{\pm} = \frac{\pm 1}{(b^{2} + 4c)^{1/2}} \left(\frac{b}{1 + e^{2i\gamma}} - \frac{b \mp (b^{2} + 4c)^{1/2}}{2} \right), \quad \tau^{-1} = 4\cos^{2}\gamma, \quad m \ge 0 \text{ integer}.$$
(7)

$$P_{n_{1}}^{+} = 0 \text{ and } P_{n_{2}}^{-} = 0 \text{ imply}$$

$$f^{+} = \frac{1}{2\cos\gamma} \frac{\sin[(n_{1}-1)\gamma]}{\sin(n_{1}\gamma)},$$

$$f^{-} = \frac{1}{2\cos\gamma} \frac{\sin[(n_{2}-1)\gamma]}{\sin(n_{2}\gamma)}.$$
(8)

For fixed γ, b, c , one looks for the smallest positive integers n_1 and n_2 such that (8) holds. If neither condition is satisfied, the corresponding γ, b, c values are called generic. In this case SAH (τ, b, c, N) is semisimple and the Bratteli diagram describing the sequence of algebras formed by taking consecutive values of N is a Pascal triangle [Fig. 1(a)]. The Bratteli diagram encodes basic in-1972 formation about the irreps of the algebra. Each point at floor N represents a distinct irrep of SAH (τ, b, c, N) whose dimension is the binomial coefficient attached to this point. If only one condition is satisfied, and without loss of generality let it be the f^+ condition, then the Pascal-triangle description is correct only for SAH $(\tau, b,$ c, N) with $N \le n_1 - 1$. For higher N values SAH $(\tau, b,$ c, N) is not semisimple, but it is possible to define a certain quotient of it which is again semisimple. Its Bratteli diagram is given by a cut Pascal triangle. One draws a vertical line through column $n_1 - 1$ and deletes all points which are strictly to the right of it. The dimension of a remaining irrep at floor N is obtained inductively from the sum of the dimensions associated with the remaining points at the (N-1)th floor (the floor above) which are connected to it. If both conditions in (8) are satisfied, and without loss of generality $n_1 \le n_2$, then the previous quotient is not semisimple for $N \ge n_2$. Again we define a new semisimple quotient whose Bratteli diagram is given by a double-cut Pascal triangle. The second cut is done by drawing a vertical line through the column $-(n_2-1)$ and deleting all points which lie strictly to the left. The "double-cut" quotients of SAH (τ, b, c, N) will be denoted SAH (n_1, n_2, N) [an example with $n_1 = n_2 = 3$ is shown in Fig. 1(b)]. Their irreps appear in critical lattice models whose continuum limit is described by minimal conformal field theories.

We now discuss the solutions of (8). If b=0, we get $f^+=f^-=\frac{1}{2}$. This case was discussed extensively in [7] and it leads to SAH(n,n,N). For $b\neq 0$, f^{\pm} depend on b and c only through the combination c/b^2 . Setting $4c/b^2 = -1/\cos^2\eta$, (8) gives

$$\eta + p_1 \pi = n_1 \gamma, \quad \eta + p_2 \pi = -n_2 \gamma, \quad p_1, p_2 \text{ integers}.$$
 (9)

The spin- $\frac{1}{2}$ xxz chain.—

$$H_{xxz} = \sum_{j=1}^{N} V_{j}(\gamma)$$

= $\sum_{j=1}^{N} \frac{1}{2} \left[\sigma_{j}^{x} \sigma_{j+1}^{x} + \sigma_{j}^{y} \sigma_{j+1}^{y} + \cos \gamma (1 - \sigma_{j}^{z} \sigma_{j+1}^{z}) + i \sin \gamma (\sigma_{j}^{z} - \sigma_{j+1}^{z}) \right],$ (10)

where σ_i^a (a=x,y,z) acts as a standard 2×2 Pauli matrix at position *i* in the chain, and as a unit matrix at all other positions. A representation of FTLJ($(4\cos^2\gamma)^{-1}$, N-1) is given by $e_i(\gamma) = (2\cos\gamma)^{-1}V_i(\gamma)$ $(i=1,\ldots, N-1)$. Setting

$$\sigma_{N+1}^{x} \pm i\sigma_{N+1}^{y} = e^{\pm i\phi}(\sigma_{1}^{x} \pm i\sigma_{1}^{y}), \quad \sigma_{N+1}^{z} = \sigma_{1}^{z}, \quad (11)$$

we obtain a family of TBC for the model (in [1,2] toroidal or twisted BC), parametrized by the angle ϕ . Under (11), the Nth term in (10) becomes

$$V_{N,1}(\gamma,\phi) = e^{i\phi\sigma_1^2/2} \frac{1}{2} \left[\sigma_N^x \sigma_1^x + \sigma_N^y \sigma_1^y + \cos\gamma(1 - \sigma_N^z \sigma_1^z) + i\sin\gamma(\sigma_N^z - \sigma_1^z)\right] e^{-i\phi\sigma_1^2/2}.$$
 (12)

Adjoining $e_N(\gamma,\phi) = (2\cos\gamma)^{-1}V_{N,1}(\gamma,\phi)$ to FTLJ $(\tau, N-1)$, we get a representation of PTLJ (τ,N) . We will proceed to analyze it according to the previous discussion. In this case an invertible \vec{C}_1 exists:

$$\vec{C}_1 = e^{i\phi\sigma_1^2/2} s_1 s_2 \cdots s_{N-1}, \qquad (13)$$

where $s_i = V_i(\pi) + 1$ permutes the *i* and *i* + 1 spaces in the tensor product space $(C^2)^{\otimes N}$ on which H_{xxz} acts. One can verify that (13) satisfies (2) using $s_i \sigma_i^a = \sigma_{i+1}^a s_i$. Physically, $\vec{C_1}$ is the translation operator of the xxz chain and the statement that the model is translation invariant is equivalent to the statement that H_{xxz} and $\vec{C_1}$ commute. Having an invertible $\vec{C_1}$ we obtain from (4) an affine

Hecke generator:

$$x_N = g_{N-1}^{-1}(\gamma) \cdots g_1^{-1}(\gamma) e^{i\phi\sigma_1^2/2} s_1 \cdots s_{N-1}.$$
 (14)

Next we would like to show that (14) is related to SAH (τ, b, c, N) . First one checks that $\hat{S}^z = \frac{1}{2} \sum_{i=1}^{N} \sigma_i^z$ commutes with the xxz PTLJ (τ, N) representation. It turns out that (14) satisfies (6) only after choosing a definite \hat{S}^z sector. In the standard spin basis where all the σ_i^z are diagonal, an \hat{S}^z sector is spanned by all basis states with a fixed number $0 \le n \le N$ of down spins. Omitting the details of the computations involved we find

$$b = e^{-i\phi/2}e^{-(N-n)i\gamma}(-1)^{n+1} + e^{i\phi/2}e^{-ni\gamma}(-1)^{N+n-1},$$

$$c = -e^{-iN(\gamma+\pi)}.$$
(15)

Inserting the last result into (9) we get

$$2\gamma(q-n_1) + \phi + \pi(2p_1 - N) = 0, \qquad (16a)$$

$$2\gamma(q+n_2) + \phi + \pi(2p_2 - N) = 0, \qquad (16b)$$

where q is the eigenvalue of \hat{S}^z in the sector under consideration. Relations (16) are the key to the understanding of the xxz degeneracies. I will not discuss them in detail but just stress the basic point. For generic values of ϕ and γ (16a) and (16b) are not satisfied and the irreps of PTLJ(τ ,N) coincide with the \hat{S}^z sectors. For ϕ, γ values for which one or both of relations (16) are satisfied, the smaller irreps of the SAH(τ, b, c, N) quotients reveal their presence. These irreps correspond to the projected systems of [1]. In [8], a neat formula was worked out for the dimensions of the projected sectors. The match between these numbers and the dimensions of the irreps of SAH(n_1, n_2, N) gives a nontrivial check of the picture described above.

The Potts chains.— The critical q-state Potts quantum chain Hamiltonian is given for a chain of L sites by

$$H_{q-\text{Potts}} = -\sum_{m=1}^{L} U_{2m-1} - \sum_{m=1}^{L} U_{2m}$$
$$= -\sum_{m=1}^{L} \sum_{k=0}^{q-1} \alpha_m^k - \sum_{m=1}^{L} \sum_{k=0}^{q-1} R_m^k R_{m+1}^{q-k} , \qquad (17)$$

where Ω_m, R_m have nontrivial action only at site *m*. They satisfy

$$\Omega_m R_m = \omega^{-1} R_m \Omega_m, \quad \Omega_m R_m^{\dagger} = \omega R_m^{\dagger} \Omega_m,$$

$$\Omega_m^q = R_m^q = 1, \quad \omega \equiv e^{2\pi i/q}.$$
(18)

A representation of FTLJ(1/q, 2L-1) is provided by taking $e_i = (1/q)U_i$ $(i = 1, \dots, 2L-1)$. Defining [2]

$$R_{L+1} = \omega^{\tilde{q}} R_1, \quad \tilde{q} = 0, \dots, q-1,$$
 (19)

we obtain a family of TBC parametrized by the integer \tilde{q} . $e_{2L} = (1/q)U_{2L}$ together with FTLJ(1/q, 2L-1) gives a representation of PTLJ(1/q, 2L) for all \tilde{q} . Equation (17) with the TBC (19) has a global Z_q invariance, namely, the associated PTLJ(1/q, 2L) representation commutes 1973 with $\hat{Z}_q = \prod_{m=1}^{L} \Omega_m$. Other TBC possessing other discrete global symmetries are also possible. I now analyze in detail the PTLJ(1/q,2L) representations for q = 2,3.

q=2.— This is the Ising model and for this case (19) is the most general TBC. Working with FTLJ($\frac{1}{2}$, 2L-1) we consider two solutions of (6): $x_{2L}=1$, R_L . According to the previous general discussion the strategy is to check whether $\overline{C}_1 x_{2L}$ satisfies (2). In fact, we only have to check it for i=2L-1. We find

$$\bar{C}_1 \Omega_L \bar{C}_1^{-1} = \hat{Z}_2 R_1 R_L \,. \tag{20}$$

This is the desired result up to the \hat{Z}_2 factor on the rhs. This factor commutes with the PTLJ($\frac{1}{2}$,2L) representations. Using the projections $\hat{Z}_2^{\pm} = \frac{1}{2} (1 \pm \hat{Z}_2)$ we obtain

$$(\hat{Z}_{2}^{\pm}\bar{C}_{1})(\hat{Z}_{2}^{\pm}\Omega_{L})(\hat{Z}_{2}^{\pm}\bar{C}_{1}^{-1}) = \pm (\hat{Z}_{2}^{\pm}R_{1}R_{L}), \qquad (21)$$

that is, $x_{2L} = 1$ yields an invertible \vec{C}_1 in the (\hat{Z}_2, \tilde{q}) representations (1,0) and (-1,1). This solution corresponds to SAH(1,3,2L). Consider now $x_{2L} = R_L$:

$$\bar{C}_1 R_L \Omega_L R_L \bar{C}_1^{-1} = -\hat{Z}_2 R_1 R_L \,. \tag{22}$$

However, since R_L anticommutes with \hat{Z}_2 , the projection of $\bar{C}_1 R_L$ on the \hat{Z}_2 sectors is zero, and as a result we do not get an invertible \vec{C}_1 . Instead, by taking $(\bar{C}_1 R_L)^2$, we get, after projecting on the \hat{Z}_2 sectors, an invertible \vec{C}_2 . This accounts for the (1,1) and (-1,0) representations. The relevant quotient here is SAH(2,2,N). Although the Bratteli diagram at N=2L has only one irrep, \vec{C}_2 belongs to a proper subalgebra of SAH(2,2,N), and this irrep splits into two, when viewed as a representation of the subalgebra. It is interesting to note that in the q-Potts representation \vec{C}_2 acts as a translation operator, while \vec{C}_1 is related to the duality transformation.

q=3.— Besides (19) we also consider the TBC [1]:

$$R_{L+1} = \omega^{\hat{q}} R_1^2, \quad \hat{q} = 0, 1, 2.$$
(23)

The resulting e_{2L} does not commute with \hat{Z}_3 but does commute with a $\hat{Z}_2(\hat{q})$ operator, $\hat{Z}_2(\hat{q}) = \hat{Z}_3^{\hat{q}} \hat{Z}_2 \hat{Z}_3^{-\hat{q}}$, where \hat{Z}_2 is defined through its action:

$$\hat{Z}_2 \Omega_m \hat{Z}_2^{-1} = \Omega_m^2, \quad \hat{Z}_2 R_m \hat{Z}_2^{-1} = R_m^2, \quad \hat{Z}_2^2 = 1.$$
 (24)

Altogether we have nine representations of PTLJ($\frac{1}{3}$,2L) labeled by (\hat{Z}_{3}, \tilde{q}) and six representations labeled by ($\hat{Z}_{2}(\hat{q}), \hat{q}$). However, $\tilde{q} = 1$ representations are equivalent to $\tilde{q} = 2$ representations through conjugation with \hat{Z}_2 and $\hat{q} = 0, 1, 2$ representations are equivalent to each other through conjugation with \hat{Z}_3 . For q = 3 one finds

$$\bar{C}_1 \Omega_L^k \bar{C}_1^{-1} = \hat{Z}_3^k R_1^k R_L^{q-k}.$$
⁽²⁵⁾

So after projecting on the three \hat{Z}_3 sectors, the solution $x_{2L} = 1$ accounts for the three (\hat{Z}_3, \tilde{q}) representations $(1,0), (\omega,1), \text{ and } (\omega^2,2), \text{ and, in each of them, an invertible <math>\vec{C}_i$ exists. This x_{2L} is related to SAH(1,5,2L). The other (\hat{Z}_3, \tilde{q}) representations are coming from $x_{2L} = \hat{Z}_2 R_L$ which is related to SAH(3,3,2L) [Fig. 1(b)]. Here only an invertible \vec{C}_2 exists. Finally, the two $\hat{q} = 0$ representations are accounted for by $x_{2L} = 1 + \hat{Z}_3 R_L + \hat{Z}_3^2 R_L^2$ which is related to SAH(2,4,2L). We get

$$\bar{C}_1 x_{2L} \Omega_L x_{2L}^{-1} \bar{C}_1^{-1} = R_1^2 R_L^2, \qquad (26)$$

that is, we get a \vec{C}_1 even without further projecting on the \hat{Z}_2 sectors.

Combining the analysis of the q-Potts models with that of the xxz model, it is possible now to account for the overlap in their spectra by identifying common irreps.

I would like to thank Uwe Grimm, Vladimir Rittenberg, and Gunter Schütz for explaining to me the details of their previous work; I especially acknowledge extensive and useful discussions with Gunter Schütz. This work was supported by the Deutsche Forschungsgemeinschaft.

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