Ehrenfest Theorem for Nonlinear Klein-Gordon Solitary Waves

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A theorem which describes the Newtonian dynamics of the center of mass (as defined with respect to the energy density) of nonlinear Klein-Gordon-type solitary waves is proved. It appears as a generalization of Ehrenfest's theorem in quantum mechanics, which describes the Newtonian dynamics of the center of mass (as defined with respect to the probability density) of the linear wave functions. As an example, the interaction of recently discovered two-dimensional pulsons [cf. E. M. Maslov, Phys. Lett. A **151**, 47 (1990)] is considered.

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The existence of stable solitary waves in nonlinear dynamical systems has put a new emphasis on the wave-particle duality in scalar field theory. This familiar concept was historically an *ad hoc* assumption made in the early 1920s by de Broglie and led to the well-known Schrödinger equation. The wave-mechanical picture of quantum mechanics arose: The particle concept became merged with the linear wave-function profile Ψ as a consequence of Born's principle postulating that the particle probability of presence is proportional to $|\Psi^2|$. One major link with the classical "macroscopical" mechanics was performed by Ehrenfest's theorem [1]: The dynamics of the center of mass of the particle wave function, defined as $\overline{\mathbf{X}}=\int_{R^3}\mathbf{X}|\Psi^2|d\mathbf{X}$, is Newtonian in the presence of an external potential V, in accordance with the averaged force $\overline{\mathbf{F}}=-\int \mathrm{grad}V|\Psi^2|d\mathbf{X}$.

In nonlinear physics, the bound dynamical problem consisting of a nonlinear solitary wave (NSW) driven by an external force and/or confined in an external potential $V(x,y,z)$ was considered by several authors [2-11]. These works emphasized the classical (Newtonian) dynamical behavior of such perturbed NSW's, but, at the same time, they exhibited the limits of considering NSW's as pure Newtonian point particles [5-7]. In some sort, NSW's occupy, in the present state of the art, an original position in scalar field theory, as they display some very peculiar properties of quantum objects —mostly due to their wave-particle duality [10,11] —together with many basic properties of classical (Newtonian) point mechanics [2-9].

In this particular context, the present Letter displays a self-consistent dynamical picture of NSW mechanics in the case of a Klein-Gordon- (KG-) type Hamiltonian density of the scalar field, which is defined as follows:

$$H(u,u_x,u_y,u_z,u_t)=\tfrac{1}{2}(u_t^2+u_x^2+u_y^2+u_z^2)+[1+\mu V(x,y,z)]U(u) \qquad (1)$$

(subscripts stand for partial derivations, as usual). It is assumed that the nonlinear wave potential $U(u)$ describes a *solitary* wave. We state this assumption precisely as follows:

$$\int_{R^3}U(u)dx\,dy\,dz<+\infty\,,$$
$$\int_{R^3}H(u,u_x,u_y,u_z,u_t)dx\,dy\,dz=\text{const}<+\infty\,. \qquad (2)$$

Note that assumptions (2) imply $\lim_{x,y,z\to\pm\infty}u_{x,y,z}=0$. The partial differential equation (PDE) corresponding to the Hamiltonian density (1) reads

$$u_{tt}-u_{xx}-u_{yy}-u_{zz}+[1+\mu V(x,y,z)]U_u=0\,. \qquad (3)$$

In Eqs. (1) and (2), μ is a scaling constant. From the continuity equation,

$$\frac{d}{dt}H-\mathrm{div}[u_t\,\mathrm{grad}u]=0\,, \qquad (4)$$

and from the definition of the scalar field momentum $\mathbf{\Pi}$ as the impulse of the center of mass of the scalar field u according to the Hamiltonian density (1):

$$\mathbf{\Pi}=\frac{d}{dt}\int_{R^3}\mathbf{X}H\,dx\,dy\,dz\,, \qquad (5)$$

we obtain $\mathbf{\Pi}=-\int_{R^3}u_t\,\mathrm{grad}u\,dx\,dy\,dz$ and hence [cf. Eq. (2)]

$$\frac{d}{dt}\mathbf{\Pi}=-\int_{R^3}u_{tt}\,\mathrm{grad}u\,dx\,dy\,dz\,. \qquad (6)$$

By appropriate integrations by parts, all six integrals of the type $\int_{R^3}u_xu_{yy}\,dx\,dy\,dz$ vanish because of the solitary-wave assumption (2). Hence we obtain from Eqs. (3) and (6),

$$\frac{d}{dt}\mathbf{\Pi}=\int_{R^3}[1+\mu V(x,y,z)]\mathrm{grad}U(u)dx\,dy\,dz\,, \qquad (7)$$

which leads to the following final result

$$\frac{d}{dt}\mathbf{\Pi}=-\mu\int_{R^3}U\,\mathrm{grad}V\,dx\,dy\,dz\,. \qquad (8)$$

Equation (8) is the equation of motion of the NSW scalar field (1) and (2) in accordance with definition (5) of the scalar field dynamical variable $\mathbf{\Pi}$. It may be regarded as an extension of Ehrenfest's theorem to nonlinear Klein-Gordon solitary waves: In the sense of a proper averaging process which only depends on the nonlinear scalar field itself, *the center of energy (mass) of this scalar field obeys an exact Newtonian dynamics.*

The small-solitary-wave-amplitude limit of Eq. (8), defined by the small-amplitude parameter $\epsilon \ll 1$, yields, of course, the original quantum-mechanical Ehrenfest theorem as recalled above. Assume $u(x,y,z,t)=\tilde{u}=\epsilon e^{i\Omega t}F(X,Y,Z,T)+\text{c.c.}$, where we introduce the stretched coordinates $X=\epsilon x$, $Y=\epsilon y$, $Z=\epsilon z$, and $T=\frac{1}{2}\epsilon^2 t$. Here the "stroboscoping frequency" Ω is of order unity and will be defined below. We assume $U(u)=pu^2+qu^4+\cdots$. The substitution of the ansatz $\tilde{u}$ in the Lagrangian density $L=u_t^2-H$, and the subsequent averaging over the rapidly oscillating terms proportional to $e^{\pm 2in\Omega t}$, $n=1,2,\ldots$, yields the corresponding average Lagrangian *à la* Whitham, whose Euler equations restitute the following nonlinear Schrödinger approximation of the original PDE (3), provided that we assume $V\sim\epsilon^2$:

$$i\sqrt{2p}F_T+F_{XX}+F_{YY}+F_{ZZ}+2p\mu(V/\epsilon^2)F+12q|F|^2F=0\,.$$

This PDE still describes the propagation of solitary waves—of vanishing amplitude, i.e., of extending width—except when the singularity in the field amplitude develops, due to the 3+1 field dimensions. Then returning to the original PDE (3) avoids it. It is crucial to note that this transformation is possible only if $\Omega^2=2p$. Then the same procedure as above, applied to the Hamiltonian (1), yields, to the *lowest ϵ^2 order, $H=4p\epsilon^2FF^*$, while we have, obviously, to the same order, $U(u)=2p\epsilon^2FF^*$.* Therefore, assuming $\mu=2$ allows Eq. (8) to reduce to the quantum-mechanical Ehrenfest theorem.

We define the case of a weak spatial modulation (of WKB type) of the nonlinear wave potential $U(u)$ in our dimensionless units as

$$V\equiv V(\epsilon x,\epsilon y,\epsilon z)\;\;(\epsilon\ll 1),\;\;\int_{R^3}U(u)dx\,dy\,dz\sim\text{const} \tag{9}$$

(note that the NSW characteristic width is of order unity). In this case, both the potential gradient grad$V$ and the space variable $\mathbf{X}$ may be approximated as constant over the NSW width defining the rapid variation of the wave potential U as well as of the Hamiltonian density H. Hence we obtain from Eqs. (5) and (8):

$$M\frac{d^2\mathbf{X}_S}{dt^2}=-\text{grad}V|_{\mathbf{X}=\mathbf{X}_S}+o(\epsilon^2)\,,$$

where

$$M=\int_{R^3}H\,dx\,dy\,dz,\;\;\mu^{-1}=\int_{R^3}U(u)dx\,dy\,dz \tag{10}$$

[cf. assumptions (2)]. Therefore the WKB case (9) of the scalar field equation of motion (8) implies the *approximated NSW point-particle Newtonian mechanics in the external potential V.* We recover the classical one-dimensional sine-Gordon $[U(u)=1-\cos u]$ kink mechanics [7,8]. Note that the field mass M which enters the equation of motion (10) is a *renormalized mass* which takes into account the presence of the potential V in the Hamiltonian density (1). We recover the conclusion of Ref. [10] which shows that the sine-Gordon "particle" oscillating in a (harmonic) confining potential according to Newton's dynamics is defined by the *static solution* of the PDE (3) (i.e., $u_{tt}\equiv 0$), and *not* by the unperturbed soliton profile.

Recently, very interesting two-dimensional so-called "pulsons" have been predicted and numerically checked in the following case [12]:

$$V\equiv 0,\;\;U(u)=\tfrac{1}{2}u^2[m^2+\lambda(1-\ln u^2)] \tag{11}$$

(the notation corresponds to Ref. [12]). We therefore have [cf. Eq. (8)] $\mathbf{\Pi}(t)\equiv 0$. Assume an initial two-pulson configuration

$$u(x,y,0)=a_1(0)\tilde{u}_1(x,y)+a_2(0)\tilde{u}_2(x,y)\,, \tag{12}$$

where each pulson is a nodeless symmetrical (about its central axis, with the radius $\mathbf{r}$) two-dimensional localized pattern, according to the formulas of Ref. [12]:

$$u_i(x,y,t)=u_i(\mathbf{r},t)=a(t)\tilde{u}_i(\mathbf{r})=Aa(t)e^{-(\lambda/2)r^2}\quad(i=1,2)\,, \tag{13a}$$

$$A=\exp\left[\frac{1}{2}\left(\frac{m^2}{\lambda}+3\right)\right],\;\;a_{tt}=-\frac{d\mathcal{V}}{da}\,,\qquad \mathcal{V}(a)=\tfrac{1}{2}\lambda a^2(1-\ln a^2)\,. \tag{13b}$$

The solitary-wave assumption (2) and the equation of motion (6) yield

$$\frac{d}{dt}\mathbf{\Pi}=-\int_{R^3}[\ddot{a}_1a_2u_1\,\text{grad}u_2+\ddot{a}_2a_1u_2\,\text{grad}u_1]dx\,dy=0\,. \tag{14}$$

Since we consider solitary waves ($\int_{R^3}\text{grad}[u_1u_2]dx\,dy=0$), we therefore have two situations: Either (i) the two pulsons are in phase $[a_1(t)=a_2(t)=a(t)]$ or (ii) they are in opposition of phase $[a_1(t)=-a_2(t)=-a(t)]$. Equations (13a) and (13b) yield $\ddot{a}a=\lambda a^2\ln a^2$. Let us state the initial configuration (12) precisely and assume that the two pulsons are both located on the x axis and lie far from each other, according to $u_2(x,y,0)=u_1(x-\Delta,y,0)$, where $\Delta\gg\sqrt{2/\lambda}$. Equation (14) gives the expression of the force $F_{1\to 2}$ originating from pulson 1 and driving pulson 2:

$$F_{1\to 2}=-\tfrac{1}{2}\alpha\pi A^2\lambda a^2\ln a^2\Delta e^{-(\lambda/4)\Delta^2}\,, \tag{15}$$

where $\alpha=+1$ if the two pulsons are in phase and $\alpha=-1$ if they are in opposition of phase. Taking into account the positive orientation of the x axis according to the choice of $u_2(x,y,0)$, we conclude that the interpulson force is attractive when the two pulsons are in phase and repulsive when they are in opposition of phase. This qualitative property has been recently checked numerically [13].

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