Statistical Properties of Resonances in Quantum Irregular Scattering

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The close relations between statistical properties of quantum dissipative systems and scattering systems is discussed. It is conjectured that for quantum chaotic scattering the distribution of the resonance poles of the S matrix is generic and follows the predictions of the Ginibre ensemble of random non-Hermitian matrices. This phenomenon has been demonstrated on a simple example of a single particle scattered by eight randomly distributed point obstacles in three dimensions.

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Irregular scattering is nowadays one of the most interesting questions in the field of quantum chaos. It has been demonstrated, for instance [1], that the existence of classical irregular scattering leads in the quantum case to an S matrix whose structure can be described by the Dyson ensemble of random matrices [2]. It is also known that the fluctuations of the quantum cross section resemble the so-called Ericson fluctuations (discovered for nuclear systems [3]). This indicates that many overlapping resonances contribute. Also, the universal fluctuations of the conductivity in mesoscopic samples has been recently interpreted in terms of chaotic scattering [4]. It is therefore natural to ask whether one can predict something about the universal features of the distribution of the corresponding resonance poles in the complex energy plane.

It is intuitively clear that the resonance poles have much in common with the eigenvalues of a quantum dissipative system. From the physical point of view, the resonance can be considered as a two-step process. During the first step an unstable intermediate (virtual) state is excited which then decays (second step) and leads for an isolated resonance to a peak in the cross section. The correspondence with the theory of the dissipative systems becomes apparent as soon as one starts with the description of the virtual state.

In order to describe the intermediate state in the scattering process, Livšic [5] introduced a dissipative operator (the so-called Livšic matrix [6]), the *eigenvalues* of which coincide with the position of the resonance poles.

The resonances which can be obtained as the poles of the analytically continued Green's function are defined according to Livšic with the help of a "restricted" quantum dynamics. Let P be a projection on a finite-dimensional subspace of the state Hilbert space $\mathcal{H}$ and let H be the relevant quantum Hamiltonian on $\mathcal{H}$. The restricted Green's function is then defined as

$$P(H-z)^{-1}P. \tag{1}$$

The Livšic matrix $B(z)$ represents the effective dissipative and energy-dependent Hamiltonian describing the quantum dynamics on the restricted state space:

$$[B(z)-z]^{-1}=P(H-z)^{-1}P \tag{2}$$

(the rest of the Hilbert space is understood as the "heat bath"). The resonances are then obtained by solving the effective eigenvalue equation

$$B(z)|\Psi\rangle=z|\Psi\rangle, \quad z\in C^{+}. \tag{3}$$

This approach is equivalent to the complex scaling method introduced later [7].

The Livšic method shows a close similarity with the treatment of the dissipative system [8]. It is therefore natural to conjecture that the statistical properties of the eigenvalues of the dissipative system and of the resonance poles in a scattering system coincide.

The question of a random matrix description of strongly damped dissipative systems has been answered by Grobe and Haake [9]. It turned out that the distribution of the Euclidean distance between neighboring complex eigenvalues of a classically chaotic dissipative system is well characterized by the predictions of the Ginibre ensemble of random matrices. Ginibre [10] dropped the requirement of Hermiticity which is imposed when dealing with the Dyson ensemble. The corresponding matrices therefore have complex eigenvalues. The statistical properties have been investigated in [11]. One of the remarkable features of the Ginibre ensemble is that it leads to cubic level repulsion; i.e., the eigenvalues strongly repel each other and are therefore correlated. This type of level repulsion should not be mixed with that of isolated resonances. For such nonoverlapping resonances the correlations between the positions and the widths can be treated separately. In the analysis of Grobe and Haake [9,11] these special correlations along the real axis, which are for positions of the Gaussian-orthogonal-ensemble type, are excluded. One considers only poles separated from the real axis by one mean pole spacing, which raises the question of a distribution in a plane rather than on a line.

In contradiction to the chaotic case, the eigenvalues of an integrable dissipative system are described as a Poisson process in the plane which yields a linear repulsion of the complex levels. So, one can state that the statistical behavior of the quantized dissipative system exhibits similar generic features as the level statistics of conservative systems (see Table I).

Let us come back to the distribution of the resonance

TABLE I. Universal features.

Classical system	Level-spacing distribution
Conservative, integrable	Poisson process on a line
Conservative, chaotic with time-reversal symmetry	Gaussian orthogonal ensemble
Dissipative, integrable	Poisson process in the plane
Dissipative, chaotic	Ginibre ensemble

poles. As already mentioned, there is a close correspondence between these poles and the eigenvalues of open quantum systems. We expect therefore a similar statistical behavior as discovered for the dissipative case.

In order to demonstrate this phenomenon we have investigated the three-dimensional elastic scattering of a quantum particle on a cluster of eight point obstacles placed randomly in the space (a "randomly deformed cube"). The important feature of the scattering system is the absence of symmetry. In the case of a symmetric cluster it would be necessary—as for bound states—to take into account only resonances which belong to the same irreducible representation of the underlying symmetry group. Consequently, any symmetric cluster (e.g., a cube) would diminish the number of available resonances appreciably.

The corresponding quantum Hamiltonian can be formally written as

$$H=-\Delta+\sum_{n=1}^{8}\alpha_n\delta(\mathbf{x}-\mathbf{x}_n)\,, \tag{4}$$

where Δ denotes the Laplacian in $L^2(R^3)$ and $\mathbf{x}=(x,y,z)$. $\mathbf{x}_n$ is the position of the nth scatterer. The potential can be considered as the first approximation (s waves only) of a partial-wave representation of a cluster of nonoverlapping potentials [12,13]. The scattering length a_n describing the single scatterer is, via the phase shift η_{0n}, related to the coupling constant α_n by

$$\cot(\eta_{0n})=\frac{1}{\sqrt{E}\,a_n}=\frac{4\pi\alpha_n}{\sqrt{E}}\,, \tag{5}$$

where E is the particle energy.

It is well known that one can describe the formal Hamiltonian (4) in a mathematically sound way using the self-adjoint extension theory (see [14] for more details). It has been demonstrated that in the case of integrable billiards the presence of the point scatterer leads to wave chaos [15]. The scattering process described by (4) can be solved almost analytically [14]. For the scattering amplitude one finds

$$f_E(\mathbf{n}_{\rm in},\mathbf{n}_{\rm out})=\frac{1}{4\pi}\sum_{l,m=1}^{8}[\Gamma(E)]^{-1}_{lm}\times\exp[i\sqrt{E}\,(\mathbf{x}_l\mathbf{n}_{\rm in}-\mathbf{x}_m\mathbf{n}_{\rm out})]\,. \tag{6}$$

Here $\mathbf{n}_{\rm in,out}$ are the unit vectors pointing in the incoming and outgoing directions and Γ is an 8×8 matrix with elements

$$[\Gamma(E)]_{lm}=\left(\alpha_l-\frac{i\sqrt{E}}{4\pi}\right)\delta_{lm}-\frac{\exp[i\sqrt{E}\,|\mathbf{x}_l-\mathbf{x}_m|]}{4\pi|\mathbf{x}_l-\mathbf{x}_m|}(1-\delta_{lm})\,. \tag{7}$$

The resonances correspond to the poles of the scattering amplitude and can be found by solving the equation

$$\det[\Gamma(E)]=0 \tag{8}$$

in the lower complex energy plane.

Equation (8) has been solved numerically on a PC for eighty statistically independent clusters $\{\mathbf{x}_n\}$ of scatterers. Each scattering center is characterized by the same scattering length $a_n=1$. The coordinates (x,y,z) of the eight obstacles have been randomly chosen between 0 and 1.

In order to investigate the local statistical properties of the poles, one has to unfold the numerical results in such a way that the mean density of the poles is equal to 1. Because the number of resonances in the considered energy interval for a given configuration is rather small (about 90), we calculate the mean density of the poles from the ensemble average. As one can see from Fig. 1 the distribution of poles of the ensemble has a rather wildly behaved density. Therefore the unfolding procedure is of crucial importance. Denoting the distance to the next neighbor by $d_i(1)$, the unfolded level spacing is determined by

$$s_i=d_i(1)\bar{\rho}_i^{1/2}\,, \tag{9}$$

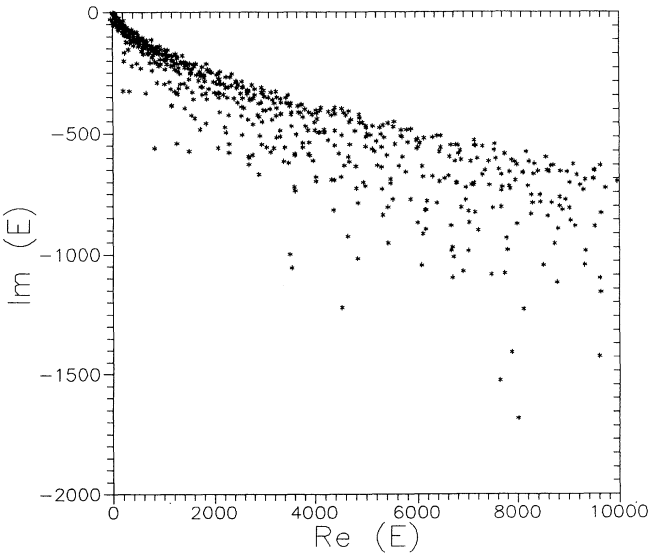

FIG. 1. Position of the resonance poles of eight different cluster configurations.

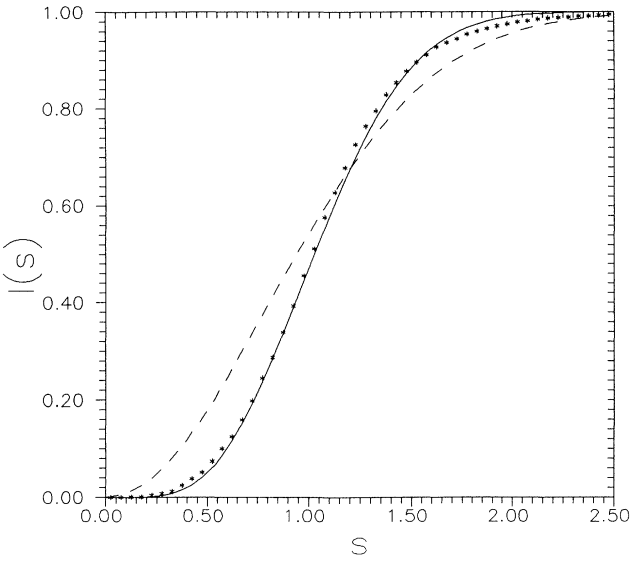

FIG. 2. Cumulative level-spacing distribution ($N=3$, see the text). The distribution has been first calculated for each cluster configuration separately and then put together. Solid line: prediction of the Ginibre ensemble; dashed line: Poisson process in the plane (uncorrelated poles); stars: calculated resonance-spacing distribution.

where the local density $\bar{\rho}_i$ is the inverse of the mean area per resonance in the complex plane around the resonance E_i. The local density has been calculated in the following way. We denote the N nearest neighbors of E_i by $E_i(1),\ldots,E_i(N)$. The area A_i covered by these neighbors has been obtained by

$$A_i^{(N)}=a_i(N)b_i(N)\,, \tag{10}$$

with

$$a_i(N)=\max \mathrm{Re}\{E_i;E_i(1),\ldots,E_i(N)\} - \min \mathrm{Re}\{E_i;E_i(1),\ldots,E_i(N)\}\,, \tag{11}$$

$$b_i(N)=\max \mathrm{Im}\{E_i;E_i(1),\ldots,E_i(N)\} - \min \mathrm{Im}\{E_i;E_i(1),\ldots,E_i(N)\}\,. \tag{12}$$

The local density then follows as

$$\bar{\rho}_i=N/A_i^{(N)}\,. \tag{13}$$

Now the level-spacing distribution can be compared with the theoretical predictions. For this purpose the cumulative distribution

$$I(s)=\int_0^s ds' P(s') \tag{14}$$

was found to be well suited because $I(s)$ depends only smoothly on the bin size of s. In the last formula $P(s)$ is the next-neighbor distribution which looks like

$$P_P(s)=\tfrac{1}{2}\pi s\exp(-\tfrac{1}{4}\pi s^2) \tag{15}$$

for the Poisson process in the plane while for the Ginibre ensemble we find (strictly speaking for ensembles of 2×2 matrices)

$$P_G(s)=2(\tfrac{9}{16}\pi)^2 s^3\exp(-\tfrac{9}{16}\pi s^2)\,. \tag{16}$$

The calculated cumulative level-spacing distribution is plotted in Fig. 2 and compared with the predictions of (15) and of (16). A clear level repulsion [cubic for $P(s)$] for small spacings and an overall agreement with the Ginibre ensemble can be observed. We want to note that for the same scattering ensemble the S matrix is, for fixed energy, well described by the Dyson circular orthogonal ensemble [13].

As already mentioned, in the case of integrable scattering systems the poles are expected to have similar statistical properties as a Poisson process in a plane. This assumption has been checked for scattering on a two-dimensional polar symmetric δ-shell potential. The results [16] support the expectations.

In summary, we have demonstrated that the resonance poles of a chaotic quantum scattering system possess a generic next-neighbor distribution which is described by the Ginibre ensemble of random matrices. We believe that the generic cubic pole repulsion can also be found in other scattering systems.

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