Discrete Versions of the Painlevé Equations

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We present discrete forms of the Painlevé transcendental equations $P_{\rm III}$, $P_{\rm IV}$, and $P_{\rm V}$ that complement the list of the already known $P_{\rm I}$ and $P_{\rm II}$. These, most likely integrable, nonautonomous mappings go over to the usual Painlevé equations in the continuous limit, while in the autonomous limit we recover discrete systems that belong to the integrable family of Quispel *et al.* Finally, we show that the discrete Painlevé mappings satisfy the same reduction relations as the continuous Painlevé transcendents, namely, $P_{\rm V} \rightarrow \{P_{\rm III}, P_{\rm IV}\} \rightarrow P_{\rm II} \rightarrow P_{\rm I}$.

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Painlevé transcendents occur frequently in physical models. The Ising model is perhaps the best known among them but these transcendents also appear in several other statistical-mechanics models related to conformal field theory [1]. Another well-known application domain of the Painlevé equations is that of integrable partial differential equations (PDE's) [2]. Reductions of integrable PDE's (and sometimes also of nonintegrable ones) often lead to one of these transcendental equations, a fact that makes possible the formulation of special solutions for the equation at hand. It is in these domains that the first two "discrete" transcendents have made their appearance [3]. The aim of this paper is to derive the forms of the next three discrete Painlevé equations using the new method of singularity confinement [4].

The (continuous) Painlevé equations were discovered at the beginning of the century by Painlevé and Gambier [5]. The method used for the derivation of these transcendental equations is related to what came to be known in the past decade as singularity analysis [6]. These equations have the so-called Painlevé property, i.e., their solutions are meromorphic functions of the independent variable, or, equivalently, their (movable) singularities are just poles [7]. Their solutions were given only in the past few years. Following the pioneering work of Ablowitz and Segur [8], it was shown that the Painlevé equations can be linearized in terms of integro-differential equations, using the inverse scattering transform scheme. Recently the inverse monodromy (isomonodromy) method has been developed for the study of the Painlevé equations [9]. Another feature suggesting nice behavior is that the Painlevé equations can be written in the Hirota bilinear form [10].

The question of the existence of a discretized form of the Painlevé equations arose naturally, given the intense activity around discrete systems. While mappings were initially used as prototypes for the study of chaos, the recent trends are towards the complementary direction, that of integrability. Numerous studies have been published concerning the construction of integrable mappings and lattices (higher-dimensional mappings) [11]. Several mappings that naturally appear in physical applications have turned out to be discrete analogs of the Painlevé equations. For example, the computation of a certain partition function in a model of 2D quantum gravity led to the discrete form d-$P_{\rm I}$ of $P_{\rm I}$ [3]:

$$x_{n+1}+x_{n-1}+x_n=(an+b)/x_n+c\,. \qquad (1)$$

Its solution using the isomonodromy approach has been given in [12].

A discrete form d-$P_{\rm II}$ of $P_{\rm II}$,

$$x_{n+1}+x_{n-1}=\frac{x_n(an+b)+c}{1-x_n^2}\,, \qquad (2)$$

has also been obtained [13] in a way closely parallel to the one used for PDE's, as a similarity reduction of the discrete version of the modified Korteweg–de Vries equation.

The derivation of the discrete forms of Painlevé equations has so far been fortuitous, because there was no discrete analog of the singularity analysis method. Quite recently a "singularity confinement" method [4] was proposed relating the integrable character of discrete systems to their singularity structure. We now have, for mappings, the equivalent of the Ablowitz-Ramani-Segur (ARS) conjecture [14] for partial and ordinary differential equations (ODE's).

The implementation of the singularity confinement method is quite simple. Given a mapping, one must first find all possible ways a singularity can emerge (this step follows closely the first step of the algorithm for ODE's where one looks for all possible leading singular behaviors). The system is said to have passed the test (and is thus a candidate for integrability) if this divergence does not propagate in (discrete) time, i.e., that it remains confined. The second step is therefore to find how far it has to propagate before it has a chance to leave room for a regular behavior (this is somewhat reminiscent of the "search for resonances" in the ARS algorithm); and finally one has to verify that indeed the singularity does *not* propagate beyond that (this last step is the equivalent of the "resonance condition").

We can illustrate this procedure in the autonomous version of $d\text{-}P_{\rm II}$ [$a=0$, $c=0$ in (2)]:

$$x_{n+1}+x_{n-1}=bx_n/(1-x_n^2)\,. \tag{3}$$

Let us assume the initial conditions are such that $x_{n-1}=1$ and x_{n-2} is finite. Then x_n diverges and $x_{n+1}=-1$. The latter is still, in a sense, part of the singularity: x_{n+1} is determined independently of other values earlier than $x_{n-1}=1$, and moreover it may be a source of further problems as it is a zero of the denominator. The singularity, however, does not propagate any further: The two sources of divergences cancel each other and x_{n+2} is finite. In fact, we find $x_{n+2}=-x_{n-2}$. The conjecture claims [4] that the fine cancellations necessary for the confinement of the singularities are associated with integrability.

Quispel, Roberts, and Thompson [15] have presented a general family of autonomous integrable mappings of the form

$$f_3(x_n)\Pi-f_2(x_n)\Sigma+f_1(x_n)=0\,, \tag{4}$$

where $\Sigma=x_{n-1}+x_{n+1}$, $\Pi=x_{n-1}x_{n+1}$, and the f_i are quartic polynomials. In [4] we have shown how to derive the discrete equations $d\text{-}P_{\rm I}$ and $d\text{-}P_{\rm II}$ starting from autonomous mappings of the Quispel family. The same method can be used to derive the discrete forms of the remaining equations.

Some insight in choosing the correct f_i's is provided by the fact that we wish to obtain equations that, in the continuous limit, go over to the Painlevé equations. Introducing a lattice parameter δ we have

$$\begin{aligned}\Sigma&=2x+\delta^2x''+\mathcal{O}(\delta^4)\,,\\ \Pi&=x^2+\delta^2(xx''-x'^2)+\mathcal{O}(\delta^4)\,,\end{aligned} \tag{5}$$

and when we extract from Eq. (4) the part involving derivatives we obtain a continuous limit ($\delta\to 0$) of the form

$$x''=\frac{f_3(x)}{xf_3(x)-f_2(x)}x'^2+g(x)\,. \tag{6}$$

So if we are aiming at a specific Painlevé equation, the first thing to do is to choose f_2,f_3 in such a way as to get $f_3(x)/[xf_3(x)-f_2(x)]$ to coincide with the factor multiplying x'^2 in that equation.

In the case of $P_{\rm III}$ we have $x''=x'^2/x+g(x)$. First of all, we should point out that the continuous form of $P_{\rm III}$ we are going to work with is

$$w''=\frac{w'^2}{w}+e^z(aw^2+b)+e^{2z}\left(cw^3+\frac{d}{w}\right)\,, \tag{7}$$

obtained from the usual one [7] through the transformation $z\to e^z$ that absorbs the w'/z term. This form agrees with (6) if we simply take $f_2=0$. In that case, in Quispel's approach, f_1 and f_3 have one quadratic common factor and, assuming that this remains true when the coefficients become n dependent, the mapping takes the form

$$x_{n-1}x_{n+1}=\frac{\kappa(n)x_n^2+\zeta(n)x_n+\mu(n)}{x_n^2+\beta(n)x_n+\gamma(n)}\,. \tag{8}$$

To fix the n-dependent coefficients we will study the singularity behavior as described before. When one solves for x_{n+1} there are two possible sources of singularity for this mapping. Either x_n is a zero of the denominator $x_n^2+\beta(n)x_n+\gamma(n)$ or x_{n-1} becomes zero. In the first case, the singularity sequence is the following: x_{n+1} diverges, x_{n+2} has a finite value $\kappa(n+1)/x_n$, and x_{n+3} would in principle be proportional to $1/x_{n+1}$ and thus zero. This would lead to a new divergence. The only way out is to ask that x_{n+2} also be a zero of the appropriate denominator, so that x_{n+3} does not vanish. Expressing x_{n+2} in terms of x_n and taking into account that this must be true for both zeros x_n of $x_n^2+\beta(n)x_n+\gamma(n)$, we obtain $\beta(n)=\beta(n+2)\kappa(n+1)/\gamma(n+2)$ and $\gamma(n)=\kappa^2(n+1)/\gamma(n+2)$. Multiplying x_n by an arbitrary function of n does not change the form of Eq. (8) but only affects the coefficients. This scaling freedom allows us to take a constant value β for $\beta(n)$, resulting in $\kappa(n+1)=\gamma(n+2)$, $\gamma(n)=\gamma(n+2)$. Thus the γ's and κ's must be constants within a given parity: $\gamma(\text{even})=\kappa(\text{odd})=\gamma_+$, $\gamma(\text{odd})=\kappa(\text{even})=\gamma_-$.

In order to study the second kind of singularity, we start with x_n such that x_{n+1} vanishes [i.e., $\kappa(n)x_n^2+\zeta(n)x_n+\mu(n)=0$]. We find then that x_{n+2} has a finite value $\mu(n+1)/\gamma(n+1)x_n$ and this would lead to a divergent x_{n+3} unless the numerator also vanishes. Substituting the expression for x_{n+2} and using the fact that again this must be true for both zeros of $\kappa(n)x_n^2+\zeta(n)x_n+\mu(n)$, we obtain

$$\mu(n)=\zeta(n)\mu(n+1)/\zeta(n+2)=\mu^2(n+1)/\mu(n+2)\,.$$

The solution to these equations is straightforward: $\mu(n)=\mu_0\lambda^{2n}$ and $\zeta(n)=\zeta_{0,\pm}\lambda^n$, where $\mu_0,\zeta_{0,\pm}$ are constants, the $\pm$ sign being related to the parity of n. Note that, in that case, there is *no second kind* of singularity at all. Indeed x_{n+3} is not allowed to diverge even though $x_{n+1}=0$. (This is reminiscent of the case of continuous equations where, if a denominator appears, one must consider the values of the dependent variable that makes this denominator vanish to ascertain that this does *not* generate a singularity.)

In order to go to the continuous limit, we start with a change of the mapping variable $y_n=\lambda^{n/2}x_n$. Moreover, at the continuous limit the distinction between even and odd must disappear. We thus write $d\text{-}P_{\rm III}$ as

$$y_{n-1}y_{n+1}=\frac{\gamma y_n^2+\zeta_0\lambda^{n/2}y_n+\mu_0\lambda^n}{\lambda^n y_n^2+\beta\lambda^{n/2}y_n+\gamma}\,. \tag{9}$$

The continuum limit is obtained by letting the lattice parameter δ go to zero, while $\gamma \approx -1/c\delta^2$ and all the other constants are of order unity: $\beta \approx a/c$, $\mu_0 \approx -b/c$, $\zeta_0 \approx -d/c$. Simultaneously, one must take $\lambda \approx 1+2p\delta$, leading to Eq. (7) with e^z replaced by e^{pz}. But p can be absorbed by rescaling z and redefining a, b, c, and d; thus we recover $P_{\rm III}$.

In order to derive d-$P_{\rm IV}$ we start from the continuous $P_{\rm IV}$:

$$w''=\frac{w'^2}{2w}+\frac{3w^3}{2}+4zw^2+2(z^2-a)w+\frac{b}{w}. \tag{10}$$

Its derivative-dependent part suggests the following mapping:

$$x_{n+1}x_{n-1}+x_n(x_{n+1}+x_{n-1})=\frac{\kappa(n)x_n^3+\varepsilon(n)x_n^2+\zeta(n)x_n+\mu(n)}{x_n^2+\beta(n)x_n+\gamma(n)}. \tag{11}$$

Solving for x_{n+1} we obtain two possible sources of divergences: Either x_n is a zero of the denominator $x_n^2+\beta(n)x_n+\gamma(n)$ or x_n+x_{n-1} vanishes. In the first case, starting with x_n, we find that x_{n+1} diverges, and x_{n+2} is finite, with value $\kappa(n+1)-x_n$. This would lead to $x_{n+3}=-x_{n+2}$ and to a divergent x_{n+4}, unless x_{n+2} is a zero of $x^2+\beta(n+2)x+\gamma(n+2)$. Expressing x_{n+2} in terms of x_n, and demanding that the resulting equation is identical to $x_n^2+\beta(n)x_n+\gamma(n)$, we obtain $\beta(n+2)+\beta(n)=-2\kappa(n+1)$ and

$$\gamma(n+2)+\beta(n+2)\kappa(n+1)+\kappa^2(n+1)=\gamma(n),$$

which lead to $4\gamma(n)-\beta^2(n)=4\gamma_{0\pm}$ (a parity-dependent constant).

For the second (potentially) singular behavior we assume that $x_n+x_{n+1}=0$ and demand that, in fact, both x_{n-1} and x_{n+2} be finite. This leads to two quartic equations for x_n. Asking that these two equations are identical implies $\mu=$const, $\zeta=\zeta_0(-1)^n$, and $\varepsilon+\gamma=\rho$(const), while β must satisfy the equation $\beta(n+3)-\beta(n+2)-\beta(n+1)+\beta(n)=0$. The solution to the latter is $\beta(n)=an+b+c(-1)^n$. Thus all the n-dependent functions are fixed and d-$P_{\rm IV}$ reads

$$x_{n+1}x_{n-1}+x_n(x_{n+1}+x_{n-1})=\frac{-(an+b)x_n^3+[\varepsilon_0-\frac{1}{4}(an+b)^2]x_n^2+\mu}{x_n^2+(an+b)x_n+[\gamma_0+\frac{1}{4}(an+b)^2]}, \tag{12}$$

where $\varepsilon_0=\rho-\gamma_0$ and we have ignored the even-odd distinction.

For the continuous limit we start from (12) and take $an+b\to 2z$, $\gamma_0\approx -1/\delta^2$, $\rho\approx -4/\delta^2+\eta$ for $\delta\to 0$, with μ and η finite. The form (10) of $P_{\rm IV}$ is obtained.

In order to obtain the discrete $P_{\rm V}$ we start from a continuous form that is symmetric under the exchange $w\leftrightarrow 1-w$ [this equation is related to the usual one [7] through $w\to w/(1-w)$] and where we furthermore eliminate the first-derivative term through the transformation $z\to e^z$ [10]:

$$w''=\frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}\right)w'^2+a\frac{w}{w-1}+b\frac{w-1}{w}+ce^zw(w-1)+de^{2z}w(w-1)(2w-1). \tag{13}$$

Comparing the coefficient of w'^2 with (6) we find that the starting point must be a mapping of the following form:

$$(2x_n-1)x_{n+1}x_{n-1}-x_n(x_{n+1}+x_{n-1})=\frac{\kappa(n)x_n^3+\varepsilon(n)x_n^2+\zeta(n)x_n+\mu(n)}{\alpha(n)x_n^2+\beta(n)x_n+\gamma(n)}. \tag{14}$$

Two sources of singularities exist here also: Either the denominator vanishes or the relation $2x_nx_{n-1}=x_n+x_{n-1}$ holds. The first case is treated in a way similar to d-$P_{\rm III}$ and d-$P_{\rm IV}$: We find

$$x_{n+2}=\frac{\kappa(n+1)+x_n\alpha(n+1)}{\alpha(n+1)(2x_n-1)}$$

and then we demand that it is a zero of $\alpha(n+2)x^2+\beta(n+2)x+\gamma(n+2)$.

For the second possible kind of potential singularity x_{n+1} diverges unless $R(x_n)+x_{n-1}x_n$ vanishes, where $R(x)$ is the right-hand side of (14). Similarly, propagating backwards, we find that x_{n-2} would diverge unless $R(x_{n-1})+x_{n-1}x_n=0$. We require that these two conditions are equivalent subject to $2x_nx_{n-1}=x_n+x_{n-1}$. The solution to these singularity confinement constraints is then straightforward. Taking $\mu=$const we have $\zeta(n)+\zeta(n+1)=-4\mu$ and thus $\zeta=-2\mu+\eta(-1)^n$. Similarly, $\varepsilon(n+1)-\varepsilon(n)=\gamma(n+1)-\gamma(n)+2\eta(-1)^n$ and thus $\varepsilon-\gamma=\theta-\eta(-1)^n$ with θ a constant. Moreover, $\kappa=2\gamma+\beta+\phi(-1)^n$ and finally $\alpha+2\beta+4\gamma+\phi(-1)^n=\sigma$ with σ another constant. Putting $\beta=\rho-\alpha$ we find $\alpha=\alpha_0\lambda^{2n}$, $\rho=\rho_0\lambda^n$ (where we have not distinguished the even-odd dependence of the constants). Thus we find d-$P_{\rm V}$:

$$(2x_n-1)x_{n+1}x_{n-1}-x_n(x_{n+1}+x_{n-1})=\frac{\frac{1}{2}(\sigma-\alpha_0\lambda^{2n})x_n^3+\{\theta+\frac{1}{4}(\sigma+\alpha_0\lambda^{2n}-2\rho_0\lambda^n)\}x_n^2-2\mu x_n+\mu}{\alpha_0\lambda^{2n}x_n^2+(\rho_0\lambda^n-\alpha_0\lambda^{2n})x_n+\frac{1}{4}(\sigma+\alpha_0\lambda^{2n}-2\rho_0\lambda^n)}. \tag{15}$$

In order to obtain the continuous limit we take $\sigma\approx 1/\delta^2$, $\theta\approx -1/\delta^2+\psi$ for $\delta\to 0$, with α_0,ρ_0,μ finite. The term λ^n goes over to e^{pz} and, as in the case of $P_{\rm III}$, the factor p can be absorbed by a scaling. We thus obtain $P_{\rm V}$ as given in Eq. (13).

One interesting feature of the Painlevé equations is that they can be reduced to lower ones by a process of coalescence [7]. In fact, starting from d-$P_{\rm II}$ [Eq. (2)] one can get d-$P_{\rm I}$ [Eq. (1)] by putting $x=1+\nu X$, $a=-2A\nu^2$, $b=-4-2C\nu$, $c=4+2C\nu-2B\nu^2$, and letting $\nu\to 0$. In the same way we can get d-$P_{\rm II}$ starting from d-$P_{\rm III}$. In fact it suffices to take

$x_n=1+\nu X_n$, $\gamma=1-\nu^2$, $\beta=-2$, $\lambda=1-a\nu^2$, $\zeta_0=-2+(b+2)\nu^2$, $\mu_0=1-(b+2)\nu^2-c\nu^3$, and we recover d-$P_{\rm II}$ [Eq. (2)] at the limit $\nu\to 0$. $P_{\rm IV}$ does not reduce to $P_{\rm III}$ but rather to $P_{\rm II}$: The same is true for d-$P_{\rm IV}$. We take $x_n=1+\nu X_n$, $\gamma=1-\nu^2(1+B+An)$, $\beta=-2+\nu^2(B+An)$, $\varepsilon=-3+(B+An)\nu^2$, $\mu=1-3\nu^2-2c\nu^3$, and we recover d-$P_{\rm II}$ [Eq. (2)] at the limit $\nu\to 0$, with $a=2A$ and $b=2B-1$. Just as $P_{\rm V}$ can be reduced to both $P_{\rm III}$ and $P_{\rm IV}$ so can d-$P_{\rm V}$. Taking $x_n=X_n/\nu\lambda^n$, $\alpha_0\approx\nu^2$, $\rho_0\approx\nu$, $\mu\approx 1/\nu^2$, $\theta\approx 1/\nu$, $\eta\approx 1/\nu$, σ and ϕ finite, we recover d-$P_{\rm III}$ in the limit $\nu\to 0$. On the other hand, to recover d-$P_{\rm IV}$, we put $x_n=\nu X_n$ and take $\alpha_0=A/\nu^2$, $\rho_0=A/\nu^2+B/\nu$, $\sigma=A/\nu^2+2B/\nu+C$, $\mu\approx\nu^2$, $\eta\approx\nu$, with A, B, C, θ, and ϕ finite. Here $\lambda=1+a\nu$ allows us to obtain the linear behavior of $\beta(n)$ in d-$P_{\rm IV}$ in the limit $\nu\to 0$.

The results we presented here have amply demonstrated the power of the new integrability criterion for discrete systems. It has made possible the systematic derivation of new transcendents. Once their form is known, it will be possible to identify them when they occur in physical models. (However, one must keep in mind that transformations of the mapping variable may alter their apparent forms.) Although it is clear that the forms we have given are discrete versions of the Painlevé equations, having the correct limit and coalescence properties, nothing can be said about their uniqueness.

Several directions of research appear open at this point. Finding the Lax pairs for the systems obtained here, Eqs. (9), (12), (15), and solving the isomonodromy problem, is one of the most interesting. Higher-order equations, having the Painlevé property, are appearing in the recent literature in relation to physical models [3,16]. Although their transcendental character has not yet been established, their discretization to integrable mappings now appears within reach of our new method.

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