Do Integrable Mappings Have the Painlevé Property?

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We present an integrability criterion for discrete-time systems that is the equivalent of the Painlevé property for systems of a continuous variable. It is based on the observation that for integrable mappings the singularities that may appear are confined, i.e., they do not propagate indefinitely when one iterates the mapping. Using this novel criterion we show that there exists a family of nonautonomous integrable mappings which includes the discrete Painlevé equations $P_{\rm I}$, recently derived in a model of two-dimensional quantum gravity, and $P_{\rm II}$, obtained as a similarity reduction of the integrable modified Korteweg–de Vries lattice. These systems possess Lax pairs, a well-known integrability feature.

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The study of discrete-time integrable systems is currently the focus of an intense activity [1]. The systems considered are either lattices, i.e., partial difference equations where both the spatial and time variables have been discretized, or mappings, i.e., finite degree-of-freedom systems in discrete time. Integrable lattices are particularly interesting as their various continuous limits generate entire hierarchies of integrable partial differential equations (PDE's). Integrability for these systems is often associated with the existence of a Lax representation, a Zakharov-Shabat linearization, but also deduced from the existence of a sufficient number of integrals of motion in involution [2]. The situation is reminiscent of the status of nonlinear evolution equations in the 1970s. More and more integrable systems were being constructed, but whenever a new nonlinear PDE appeared in a physical application its integrability could be surmised only at the cost of lengthy numerical investigations. The situation was dramatically modified with the advent of the Painlevé criterion [3]. A study of the singularity structure of the equation at hand allowed one, in most cases, to make a safe prediction of its integrable (or not) character. This development became possible only after the production of a "critical mass" of integrable PDE's on which the conjecture relating integrability and singularity structure could be tested. This is roughly what is happening now with discrete-time systems. More and more integrable lattices and mappings are appearing in the literature [4] but no criterion for the integrability of a new system existed until now. So there has been no way to predict the integrability of mappings short of exhibiting Lax pairs or a set of commuting integrals, i.e., proving the integrability. The present work gives a new criterion for assessing the integrability of discrete-time systems. It is based on the study of the movable singularities of a mapping and so it is, in some way, the analog of the Painlevé criterion for continuous-time systems.

In order to introduce our method let us study the integrable lattice of potential–Korteweg–de Vries type, presented in [2,5], that we write here as

$$x_j^{i+1} = x_{j+1}^{i-1} + \frac{1}{x_j^i} - \frac{1}{x_{j+1}^i}\,. \tag{1}$$

The structure of the lattice associated with the evolution equation (1) can be easily assessed in Fig. 1. Initial data can be given on the staircase which joins the points $(i-1)$ and $(i-2)$. Evolution can be understood as taking place towards increasing i's. Now let us assume that

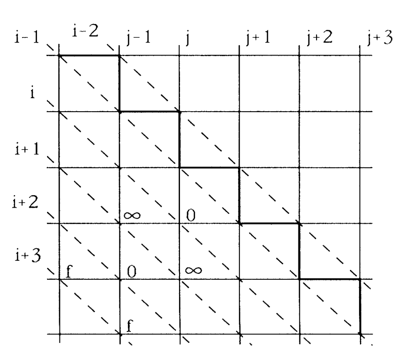

FIG. 1. Structure of the lattice for the evolution equation (1): The index j runs over the vertical lines while i labels the slanted (dashed) lines. Initial data are given on the heavily drawn line. The symbols 0, ∞, and f (which stands for "finite value") indicate the location of the corresponding values on the lattice.

during the successive applications of (1) the value of x at (i,j) becomes zero. This is not at all impossible and the point where this occurs depends on the initial data, i.e., the singularity induced is movable. From (1) it is clear that x diverges at both sites $(i+1,j-1)$ and $(i+1,j)$ and that it vanishes again at $(i+2,j-1)$. Now the crucial question is what happens at the sites $(i+3,j-2)$ and $(i+3,j-1)$. It turns out that, due to the precise form of (1), there exists a fine cancellation leading to finite values at both sites. In fact, for x_{j-1}^{i+3} we find just $x_j^{i-1}+1/x_{j-1}^{i}-1/x_j^{i+2}$ and a similar expression for x_{j-2}^{i+3}. Thus the singularity is perfectly confined. This would not have been true if the evolution did not have the form (1). In fact, modifications of the relative coefficients of the various terms in the lattice (that, presumably, destroy integrability) lead to singularities radiating out from the site (i,j) all the way to infinity. The situation is reminiscent of the difference between the singularity structures of integrable and nonintegrable continuous-time systems. Integrable systems have the Painlevé property: Their singularities are isolated and single valued; thus one can make a loop around each of them and come back to the starting point. In nonintegrable systems the singularities condense to natural boundaries [6] that one cannot cross. Thus we can see from this first example that integrability in discrete-time systems is related to confined (movable) singularities. A more stringent test to this hypothesis can be obtained if one chooses the initial conditions so as to have two adjacent zeros leading to three adjacent singularities (see Fig. 2). The pattern of singularities is now more complex but still after a second row of singularities the divergences are canceled out.

Let us see how the notion of "singularity confinement" works out in a completely solvable mapping: the discretized anharmonic oscillator. In [7] Hirota has presented the (integrable) mapping

$$[x_{n+1}-2x_n+x_{n-1}]/\delta^2=-\alpha x_n-\beta x_n^2[x_{n+1}+x_{n-1}]/2 \tag{2}$$

that corresponds to a discretization of a quartic oscillator. Another way of looking at (2) is the following: As shown recently by Yoshida [8], this mapping is the symplectic integrator of order 2 for the quartic oscillator. Hirota produced the complete solution of (2) in terms of elliptic functions: $x(t)=x_0\mathrm{cn}[\Omega(t-t_0),\kappa]$ where Ω and κ are given by $1-\mathrm{cn}(\delta\Omega)/\mathrm{dn}^2(\delta\Omega)=\alpha\delta^2/2$ and $2\kappa^2=x_0^2\beta\delta^2/[\mathrm{sn}(\delta\Omega)/\mathrm{dn}(\delta\Omega)]^2$ and the time variable is discretized $t=n\delta+t_0$. Now let us assume that for a given n $x(n)$ diverges. Using the addition formulas for the elliptic cosine we can easily verify that $x(n\pm1)=\pm i(2/\beta\delta^2)^{1/2}$ and also $x(n+2)=-x(n-2)$. Thus $x(n-1)$ has precisely the value that guarantees a divergence for $x(n)$ and $x(n+1)$ has the value that compensates this divergence. Moreover, the memory of the initial condition, that has propagated up to $x(n-2)$, survives past the singularity in $x(n+2)$.

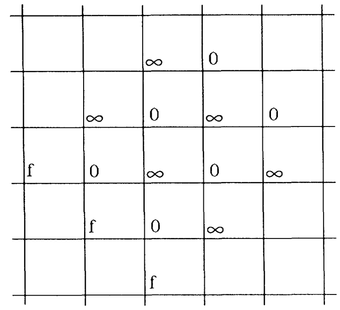

FIG. 2. Singularity pattern for initial data that lead to two adjacent zeros for Eq. (1), with the same conventions as for Fig. 1.

Thus our conjecture concerning the integrability of mappings can be stated in the following (intuitive) way: The movable singularities of integrable mappings are confined, i.e., they are canceled out after a finite number of steps. Moreover, the memory of the initial condition is not lost whenever a singularity is crossed. It goes without saying that not all mappings will be covered by our conjecture: Strictly polynomial mappings (like the Hénon map [9]) do not have movable singularities at finite distance. Moreover, one must be careful as to whether singularities are possible, within a given discrete-time system, without assuming divergent initial conditions. In the latter case the singularity is not a movable one.

Going back to Hirota's anharmonic oscillator we may remark that (2) can be written as

$$z_{n+1}+z_{n-1}=2\mu z_n/(1+z_n^2)\,, \tag{3}$$

with $z_n=(\beta\delta^2/2)^{1/2}x_n$ and $\mu=1-\alpha\delta^2/2$, i.e., the well-known McMillan mapping [10]. Expression (3) is just a particular case of a more general mapping:

$$x_{n+1}+x_{n-1}=-\frac{ax_n^2+bx_n+c}{dx_n^2+ex_n+f}\,. \tag{4}$$

As an application of our method we will deduce the constraints on the parameters $a,b,\ldots,f$ for (4) to be integrable solely from the study of its singularities. Two different cases must be distinguished at the outset: either $d\neq0$ (in which case we can take $d=1$) or $d=0$ (and we take $e=1$ for the mapping to be able to diverge). In both cases the variable x can be translated so that $f=0$. In the first case the mapping has the form

$$x_{n+1}+x_{n-1}=-\frac{ax_n^2+bx_n+c}{x_n(x_n+e)} \tag{5}$$

(provided the numerator and denominator do not have common factors). Let us assume that x_n becomes zero. Iterating the mapping we find successively $x_n=0$, $x_{n+1}=\infty$, $x_{n+2}=-e$. For x_{n+3} to be finite we obtain the condition $e=a$, which leads precisely to the only integrable form of (5). Similarly for $d=0$ we start from

$$x_{n+1}+x_{n-1}=-ax_n+b+c/x_n \tag{6}$$

and assume that $x_n=0$ and we find that $x_{n+1}=\infty$. A first condition for the nonpropagation of the singularity, i.e., for x_{n+4} to be finite, is $a(a^2-1)=0$. In the case $a=-1$ we find a supplementary condition $b=0$. Thus this last case can be transformed to $a=1$ for the new variable $z_n=(-1)^n x_n$ with $c\to -c$. Again $a=0$ and $a=1$ are the only known integrable cases.

The McMillan mapping belongs to a much more general family of integrable mappings obtained by Quispel, Roberts, and Thompson [11]:

$$x_{n+1}=\frac{f_1(x_n)-x_{n-1}f_2(x_n)}{f_2(x_n)-x_{n-1}f_3(x_n)}, \tag{7}$$

where the f_i are quartic polynomials. Integrability here is synonymous to the existence of a one-parameter family of invariant curves $F(x_n,x_{n+1})=\text{const}$. We will not discuss here whether this definition is, in general, equivalent to the existence of Lax pairs and a Zakharov-Shabat linearization. It suffices to say that in the case of the Quispel mappings the biquadratic equation $F(x_n,x_{n+1})=\text{const}$ can be parametrized in terms of elliptic functions leading presumably to the full solution [12].

The examples presented above have shown clearly that the new criterion can be applied efficiently as an integrability detector for discrete-time systems. In fact, the calculations necessary in order to control the singularity confinement can be easily performed on a computer using a program for algebraic manipulations. Still there exists one more interesting application of our method that we would like to present here. It concerns discrete Painlevé equations. A discrete version of $P_{\rm I}$ has appeared in recent works on two-dimensional quantum-gravity models [13], while in [14] Nijhoff and one of us (V.P.) have obtained a discrete version of $P_{\rm II}$ from similarity reductions of the integrable modified Korteweg–de Vries lattice [2,15]. Both these nonautonomous difference equations have Lax pairs and reduce to the usual $P_{\rm I}$ and $P_{\rm II}$, respectively, at the continuous-time limit. Let us start by generalizing the McMillan mapping (5) to the nonautonomous case (after taking $a=e=1$ by rescaling the variables):

$$x_{n+1}+x_{n-1}=-\frac{x_n^2+B(n)x_n+C(n)}{x_n(x_n+1)}. \tag{8}$$

As in the autonomous case we assume that $x_n=0$, which leads to the following condition for singularity confinement: $C(n+1)-C(n-1)-B(n+1)+B(n)=0$. Similarly starting from the second root of the denominator $x_n=-1$, we find $C(n+1)-C(n-1)+B(n-1)-B(n)=0$. Combining the two equations we obtain $B(n+1)-2B(n)+B(n-1)=0$ and $B(n)=\lambda n+\mu$. Substituting back we obtain the following for C: $C(n+1)-C(n-1)=\lambda$, and thus $C(n)=\lambda n/2+\nu$. With these expressions of B and C and with $z_n=2x_n+1$ we find

$$z_{n+1}+z_{n-1}=\frac{z_n(an+\beta)+\gamma}{1-z_n^2}, \tag{9}$$

which is precisely the discrete $P_{\rm II}$.

Similarly one can start from a nonautonomous form of (6):

$$x_{n+1}+x_{n-1}=-x_n+B(n)+C(n)/x_n. \tag{10}$$

A first condition for singularity confinement is $B(n+1)-B(n)=0$. Thus B is constant. Once this is implemented we find a second (and sufficient) condition: $C(n+3)-C(n+2)-C(n+1)+C(n)=0$. The general solution of this equation is $C(n)=an+\beta+\gamma(-1)^n$. We can remark here that the last term of $C(n)$ will disappear at the continuous limit. Here again (for $\gamma=0$ and $B=b$) we obtain

$$x_{n+1}+x_{n-1}+x_n=b+(an+\beta)/x_n, \tag{11}$$

i.e., the discrete $P_{\rm I}$.

Let us now summarize our findings. A new criterion for the detection of integrability of discrete-time systems has been derived, based on the notion of singularity confinement. This last feature is, for discretized systems, the equivalent of the Painlevé property. Its implementation is perfectly algorithmic, once one has found *all* the possible ways in which the mapping can lead to divergences (another resemblance to the Painlevé method). The singularities considered are movable, i.e., initial-condition dependent. It is essential that the memory of the initial conditions survive past the singularity. The treatment of nonautonomous and/or multidimensional systems does not present particular difficulties. It would be interesting, using this new criterion, to investigate the existence of higher transcendental equations, i.e., the analogs of $P_{\rm III}$ to $P_{\rm VI}$.

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