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Numerical Solution of the Three-Anyon Problem

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Using the Faddeev equations in configuration space, we solve numerically the problem of three anyons interacting via harmonic forces. The relevance of our results to the many-anyon problem is discussed.

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Quantum particles in two dimensions can have an arbitrary angular momentum. Therefore, they can have any statistics (anyons) interpolating between Bose statistics and Fermi statistics [1-3], the only allowed statistics in three dimensions. It has been argued that quasiparticles with fractional statistics might explain the occurrence of fractional fillings in the fractional quantum Hall effect [4,5], and account for the elementary excitations of Anderson's spin-liquid model [6] (resonating valence bond). It has been suggested that in the ideal-gas limit these excitations exhibit a new type of superconductivity which might be relevant to high-temperature oxide superconductors.

It is usually convenient to describe anyons in terms of fermions or bosons carrying a fictitious electric charge and a fictitious magnetic flux [7] interacting according to a Chern-Simons action characterized by a parameter θ [8]. The statistical phase then appears as an Aharonov-Bohm phase [9,10] of the fictitious (statistical) gauge field. This approach will be followed in this paper. Because of the statistical interaction the multiparticle wave function does not factorize, as in the free-fermion or free-boson case, and the few- or many-body problem becomes complex, even in the absence of other interactions. The two-anyon problem is still exactly solvable but the three- and many-anyon problems are not. A restricted and simple set of exact solutions for the three-anyon problem was discussed some time ago by Wu [11], and generalized recently to many-anyon systems [12]. Wu's solutions describe states in which the angular momentum of all the anyons points in the same direction and lead to a linear θ dependence of the energy. The complete set of solutions, and, in particular, the description of the fermionic ground states in terms of bosons, is not known. It is the purpose of this Letter to address this problem. We will provide an exact numerical solution to the three-anyon problem by formulating the problem in terms of the Faddeev equations [13] in configuration space. For a discussion of numerical solutions of the three-body problem we refer to Ref. [14]. For convenience we will assume that the anyons interact via harmonic forces.

When we denote the position of the ath particle in the plane by $\mathbf{r}_a$ and its conjugate momentum by $\mathbf{p}_a \equiv -i\partial/\partial\mathbf{r}_a$, the Hamiltonian for three anyons interacting via harmonic forces reads

$$H_\theta = \frac{1}{2}\sum_{j=1}^{3}\left(\mathbf{p}_j - \frac{\theta}{2\pi}\mathbf{A}_j\right)^2 + \frac{1}{2}\sum_{j=1}^{3}\mathbf{r}_j^2\,, \tag{1}$$

with

$$A_{1i} = \frac{\epsilon_{ij}(r_{1j}-r_{2j})}{|\mathbf{r}_1-\mathbf{r}_2|^2} + \frac{\epsilon_{ij}(r_{1j}-r_{3j})}{|\mathbf{r}_1-\mathbf{r}_3|^2}\,, \tag{2}$$

and $\mathbf{A}_2$ and $\mathbf{A}_3$ are obtained by a cyclic permutation. Here, the A's stand for the long-range "statistical" gauge fields. They are pure gauge and do not induce any (Lorentz) force. Throughout this paper the particles will be assumed to be bosons, and the statistical parameter θ will range from 0 to π, unless specified otherwise. In the Hamiltonian, θ is to be taken modulo 2π as can be easily seen from the statistical Chern-Simons action from which the above Hamiltonian follows in Coulomb range.

For a three-body problem it is useful to introduce Jacobi coordinates $\boldsymbol{\xi},\boldsymbol{\eta}$ and their conjugate momenta $\mathbf{p}_\xi,\mathbf{p}_\eta$,

$$\xi_1 = (1/\sqrt{2})(\mathbf{r}_2-\mathbf{r}_3),\quad \eta_1 = (\tfrac{2}{3})^{1/2}[\tfrac{1}{2}(\mathbf{r}_2+\mathbf{r}_3)-\mathbf{r}_1]\,, \tag{3}$$

with their cyclic permutations. They satisfy the relations $\xi_1+\xi_2+\xi_3=0$ and $\eta_1+\eta_2+\eta_3=0$. The sum of the Jacobi momenta is also zero.

It is possible to obtain the solutions of the Schrödinger equation $H_\theta \Psi = E\Psi$ by a straightforward diagonalization. Since the potentials are singular at the anyon locations, this approach requires a complete set of three-body wave functions that vanish rapidly enough at the anyon locations so as to overcome the singular nature of the vortices in (2). We will not follow this approach. Instead, we will use the Faddeev equations in configuration space to obtain the eigenvalues and eigenfunctions of the above Schrödinger equation.

It is useful to introduce complex coordinates

$$x_k = \xi_{kx} + i\xi_{ky}, \quad k=1,2,3\,, \tag{4}$$

$$y_k = \eta_{kx} + i\eta_{ky}, \quad k=1,2,3\,, \tag{5}$$

and the corresponding complex momenta p_{kx} and p_{ky}. The Hamiltonian (1) can be simplified considerably by using reduced three-body wave functions $\Psi_\pm$ (a similar transformation was considered in Ref. [15])

$$\Psi(1,2,3) = (x_1\bar{x}_1 x_2\bar{x}_2 x_3\bar{x}_3)^{\pm\theta/2\pi}\Phi_\pm(1,2,3)\,, \tag{6}$$

where the prefactors are of the Jastrow-Laughlin [4] type. Throughout, the three-body wave function is understood to be normalizable. (In general, normalizability of the wave function is too strong a requirement. In this case, however, this condition selects the correct solutions of the Schrödinger equation.) As a result, the reduced Hamiltonian H_+ is

$$H_+ = H_0 - \frac{2\theta}{\pi}\left[\frac{1}{\bar{x}_1}\partial_{x_1} + \frac{1}{\bar{x}_2}\partial_{x_2} + \frac{1}{\bar{x}_3}\partial_{x_3}\right], \tag{7}$$

$$H_0 = -2\partial_{x_1}\partial_{\bar{x}_1} - 2\partial_{y_1}\partial_{\bar{y}_1} + \tfrac{1}{2}(x_1\bar{x}_1 + y_1\bar{y}_1)\,, \tag{8}$$

and $H_- = H_+^\dagger$. From now on, we will use the transformation (6) and omit the subscript $\pm$. The wave functions $\Psi(1,2,3)$ will be taken to be bosonic. The fermionic formulation yields the same spectrum which can be obtained analogously.

The decomposition (6), while still very general, builds the correct boundary conditions in the anyonic wave function commensurate with the vortex configurations, provided that Φ is finite in the plane. This will be the case below. The non-Hermiticity of the reduced Hamiltonian (7) and (8) is harmless. From Eqs. (7) and (8) it can be shown immediately that ($\rho^2 = \bar{x}_1 x_1 + \bar{y}_1 y_1$)

$$\Phi_+(1,2,3) \sim \bar{x}_1^{p_1}\bar{x}_2^{p_2}\bar{x}_3^{p_3} e^{-\rho^2/2} + \text{permutations}\,, \tag{9}$$

$$\Phi_-(1,2,3) \sim x_1^{p_1} x_2^{p_2} x_3^{p_3} e^{-\rho^2/2} + \text{permutations} \tag{10}$$

are eigenstates of $H_\pm$ with energies $E_\pm = \pm 3\theta/\pi + 2 + p_1 + p_2 + p_3$. The powers p_k have to be restricted such that the eigenfunctions are single valued and normalizable. These solutions, which can be generalized by adding an arbitrary radial excitation, were first discussed by Wu [11].

The reduced wave function Φ of the three anyons can be expressed as the sum of three Faddeev components:

$$\Phi(1,2,3) = \phi(x_1,y_1) + \phi(x_2,y_2) + \phi(x_3,y_3)\,. \tag{11}$$

For identical particles the functions $\phi(j)$ with $j=1,2,3$ have the same functional form. Only their argument differs. This decomposition is completely general and allows us to find the most general solution of the Schrödinger equation. The reduced Schrödinger equation $H\Phi = E\Phi$ can be separated into three coupled equations

$$(H_0 - E)\phi(k) = -V(k)[\phi(1) + \phi(2) + \phi(3)]\,, \quad k=1,2,3\,, \tag{12}$$

where we have defined

$$V(k) = -\frac{2\theta}{\pi}\frac{1}{\bar{x}_k}\partial_{x_k}, \quad k=1,2,3\,. \tag{13}$$

These are the Faddeev equations [13] in configuration space. The sum of these three equations gives back the reduced Schrödinger equation, and it can be shown that they uniquely determine its solutions. Since the three particles are identical, the three Faddeev equations have the same structural form (modulo permutations) and only the first equation has to be solved, whereas the other solutions follow by permutation. The solution to (12) together with (6) and (11) allows for the reconstruction of the exact three-body wave function $\Psi(1,2,3)$.

We require that the eigenstates of the Schrödinger equation are *single valued*. Moreover, we impose the boundary conditions that the wave function vanishes sufficiently fast at the position of the anyons and at infinity. These conditions can be summarized by the expansions of the reduced wave function with total angular momentum j in the harmonic-oscillator eigenfunctions as follows:

$$\phi^j(x_1,y_1) = \sum_{m_1=-\infty}^{\infty}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a_{n_1 n_2}^{m_1}\phi_{n_1}^{m_1}(x_1)\phi_{n_2}^{j-m_1}(y_1)\,, \tag{14}$$

where the harmonic-oscillator wave functions ϕ_n^m are given by

$$\phi_n^m(x) = C_n^m x^m L_n^{|m|}(\bar{x}x)\exp(-\tfrac{1}{2}\bar{x}x)\,, \tag{15}$$

for positive m. For negative m the wave functions follow by exchanging x and $\bar{x}$. The normalization constants C_n^m are chosen such that the norm of ϕ_n^m is equal to 1. The requirement for physical states is that the total wave function $\Psi(1,2,3)$ is either symmetric (for bosons) or antisymmetric (for fermions) under the interchange of any pair of particles. Since the prefactors in (6) are symmetric under any interchange, the symmetries of Ψ are the symmetries of Φ. From the definition of the Jacobi variables it can be seen that these symmetries are fulfilled by restricting the sum over m_1 to be even for bosons, and to be odd for fermions.

By substituting the expansion (14) and (15) into the Faddeev equation, and projecting the equations onto $\phi_{\bar{n}_1}^{\bar{m}_1}(x_1)\phi_{\bar{n}_2}^{j-\bar{m}_1}(y_1)$ we obtain a matrix equation. In order to solve this equation by a numerical diagonalization, we truncate the basis to all states below a fixed number of harmonic-oscillator quanta. The matrix elements are calculated by transforming the complex variables to polar coordinates. The angular integrations are performed by a Gauss-Tschebychev integration [16], whereas the radial integrals are evaluated with a Gauss-Laguerre integration [16]. The number of points is chosen such that the integration is exact. This is possible because the integrands are *polynomials* in the variables $|x_1|^2, |y_1|^2$ and in the angular variables $\exp(i\arg x_1), \exp(i\arg y_1)$ which can be shown with the help of the following identities for the Jacobi variables:

$$x_2 = -\tfrac{1}{2}x_1 + \tfrac{1}{2}\sqrt{3}y_1, \quad y_2 = -\tfrac{1}{2}\sqrt{3}x_1 - \tfrac{1}{2}y_1, \quad (16)$$

$$x_3 = -\tfrac{1}{2}x_1 - \tfrac{1}{2}\sqrt{3}y_1, \quad y_3 = \tfrac{1}{2}\sqrt{3}x_1 - \tfrac{1}{2}y_1. \quad (17)$$

We have conducted several checks on our numerical results. First, for $\theta = \pi$ the exact fermionic three-particle spectrum is reproduced. Second, for small values of θ we have checked for some low-lying states that our results agree with first-order perturbation theory. Third, by performing calculations with different basis sizes (up to 18 units of the harmonic frequency) we have shown that the eigenvalues are insensitive to a truncation to a finite basis set.

Our main numerical results are shown in Fig. 1, where the behavior of the spectrum of the three-body system is shown versus the statistical flux θ up to an energy of 7. We observe two kinds of states. First, states for which the total energy changes by 3 or -3 going from $\theta = 0$ (boson) to $\theta = \pi$ (fermion). These states have an energy that depends linearly on θ, and have a simple wave-function content. They correspond to anyons rotating in concert (either up or down), and have already been noted by Wu [11]. Second, we find new states for which the total energy changes by 1 or -1 going from 0 to π. Some of these states depend quadratically on θ near 0 or π. In particular, the fermionic ground state expressed in a bosonic wave function belongs in this category. It originates from a $j = -3$ and energy of 5 bosonic state, and behaves as $4 + (\frac{5}{3})^{1/2}(\pi - \theta)^2$ for θ close to π. (A nonlinear dependence in θ has also been noted in variational calculations [17].) We also want to note the symmetry in θ of these states about the semionic point ($\theta = \pi/2$).

The angular momentum of the fermionic state ($\theta = \pi$) can be identified as $j - 3$, where j is the (conserved) angular momentum of the bosonic states. The angular momenta of the bosonic states emerging from energy 2, 4, 5, 6, and 7 are given by $[0]$, $[0, \pm 2]$, $[\pm 1, \pm 3]$, $[0^2, \pm 2, \pm 4]$, and $[\pm 1^2, \pm 3^2, \pm 5]$, respectively. The upper power refers to the multiplicities of the angular momenta.

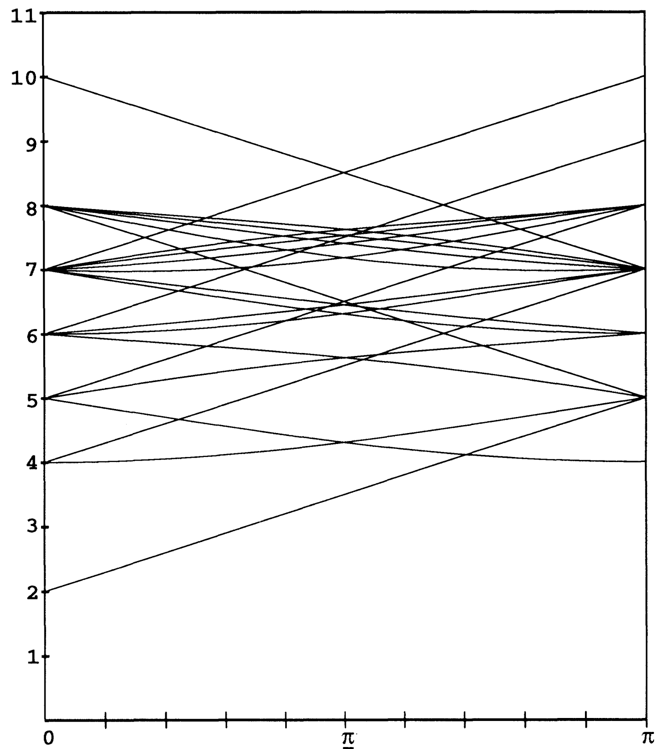

FIG. 1. The energy spectrum of three anyons interacting via harmonic forces vs the statistical flux θ. The energies are given in units of the harmonic frequency.

In Fig. 1, degeneracies occur for states with a linear θ dependence of the energy. For example, states going from 8 to 11 when θ goes from 0 to π occur for the angular momenta 0, -2^2, -4^2, and -6^2. As far as the θ dependence for small θ is concerned, we found that all slopes that occur at a given energy and angular momentum also occur at higher energies and at the same angular momentum.

In conclusion, we have obtained the spectrum for three anyons interacting via harmonic forces. Apart from the known states with a linear θ dependence we also find states with a more complicated θ dependence for which the energy changes only by one unit going from $\theta = 0$ to π. The stability of the ground-state fermion energy to variations in the statistical flux (the fermionic energy changes only by about 0.12 in the range $\theta \in [0.7\pi, \pi]$) perhaps justifies the mean-field estimates of the ground-state energy of the many-anyon problem. Since the slope of the linearly rising bosonic state grows with the number of anyons A as $A(A-1)/2$, further crossings with the ground-state level cannot be excluded.

A more comprehensive account of the results reported above, as well as an extension to the many-anyon problem, will be presented in a future publication.

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[1] J. M. Leinaas and J. Myrheim, Nuovo Cimento B **37**, 1 (1977).
[2] G. A. Goldin, R. Menikoff, and D. H. Sharp, J. Math. Phys. **22**, 1664 (1981).
[3] F. Wilczek, Phys. Rev. Lett. **48**, 1144 (1982).
[4] R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
[5] B. I. Halperin, Phys. Rev. Lett. **52**, 1583 (1984).
[6] R. B. Laughlin, Science **242**, 525 (1988).
[7] F. Wilczek, Phys. Rev. Lett. **49**, 957 (1982).
[8] For a review, see R. Mackenzie and F. Wilczek, Int. J. Mod. Phys. A **3**, 2827 (1988), or *Fractional Statistics and Anyon Superconductivity,* edited by F. Wilczek (World Scientific, Singapore, 1989).
[9] W. Ehrenberg and R. E. Siday, Proc. Phys. Soc. London B **62**, 8 (1948).
[10] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[11] Y. S. Wu, Phys. Rev. Lett. **53**, 111 (1984).
[12] J. Grundberg, T. H. Hansson, A. Karlhede, and E. Westerberg, Stockholm University Report No. USITP-91-2 (to be published); G. V. Dunne, A. Lerda, and C. A. Trungenberger, MIT Report No. MIT-CTP-1938, 1991 (to be published).
[13] L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1961) [Sov. Phys. JETP **12**, 1014 (1961)] .
[14] *Configuration Space Faddeev Equations: Numerical Methods in Models and Methods in Few Body Physics,* edited by L. S. Ferreira *et al.*, Lecture Notes in Physics Vol. 273 (Springer, Berlin, 1987); R. A. Malfliet and J. A. Tjon, Nucl. Phys. **A127**, 161 (1969); A. D. Jackson, A. Lande, and P. U. Sauer, Nucl. Phys. **A156**, 1 (1970).
[15] M. Ma and F. C. Zhang, Phys. Rev. Lett. **66**, 1769 (1991).
[16] M. Abramovitz and I. A. Stegun, *Handbook of Mathematical Functions* (U.S. GPO, Washington, DC, 1964).
[17] T. H. Hansson (private communication).