Effective String Theory

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The effective conformal field theory governing the long-distance dynamics of string solitons in four dimensions, such as the Nielsen-Olesen vortex or QCD strings, is described. It is an interacting, Poincaré-invariant conformal field theory with four bosons and $c=26$. The compatibility of these naively contradictory features is explicitly demonstrated through second order in a perturbation expansion about the long-string vacuum.

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Consider the Abelian Higgs model with spontaneous symmetry breaking at a length scale a. This theory contains a Nielsen-Olesen vortex [1] whose thickness and string tension are, up to dimensionless couplings, characterized by the same scale a. At distances much greater than a, most degrees of freedom are frozen out. Only the transverse oscillations of the vortex, which are massless by Goldstone's theorem, remain. Integrating out the heavy degrees of freedom leaves an effective field theory of the transverse oscillations,

$$\int \mathcal{D}X e^{iS} . \qquad (1)$$

Here, X denotes the world sheet swept out by the vortex in spacetime. The vortex has an energy proportional to its length, so the leading term in the action is

$$S_N = -\frac{1}{4\pi a^2} \times \text{area} , \qquad (2)$$

where the area is computed with the induced metric and higher-order corrections are relatively suppressed by powers of a.

Equations (1) and (2) appear to define something rather familiar, the bosonic string theory. However, for the purpose of describing the quantum states of a long Nielsen-Olesen vortex *none of the standard string quantizations is correct.* To see why this is so, consider the following properties of the vortex theory. First, it is Lorentz invariant with a positive Hilbert space, since these properties are inherited from the underlying field theory. Second, the vortex has only transverse oscillations, $D-2$ in all [2]. In the collective-coordinate method [3] one starts with the static classical solution, a long straight string, and the collective oscillations are the Goldstone modes associated with broken translational symmetries. The long-straight-string solution breaks the $D-2$ transverse translational symmetries, but is invariant under time and longitudinal translations. For simple examples like the Abelian Higgs model, one can check this reasoning explicitly against the spectrum of zero modes.

Now compare these features with properties of the standard string quantizations. The light-cone quantization spoils Lorentz invariance outside the critical dimension of 26 [4]. The covariant (Virasoro) quantization leads to longitudinal oscillators outside the critical dimension giving a total of $D-1$ oscillators [5]. The Polyakov quantization contains an additional Liouville mode and also leads to $D-1$ oscillators [6]. Thus none of these quantizations can apply to the Nielsen-Olesen vortex.

The same paradox holds for long QCD flux tubes, where we consider for simplicity the hypothetical case of ultramassive quarks so the tubes cannot break. Here the theory is strongly coupled so one cannot carry out the collective-coordinate method as explicitly, but a massless longitudinal mode would be unnatural from the two-dimensional point of view: Goldstone's theorem does not protect it from acquiring a mass. The only degrees of freedom required by the symmetries of the low-energy theory are the $D-2$ transverse oscillators.

One possible way to resolve this paradox would be to start with the known path integral for the underlying field theory and carry out the collective-coordinate quantization with careful attention to the path-integral measure. Then convert the path integral to covariant form, in which the integral runs over D unconstrained X^μ fields with action [7,8]

$$S_0 = \frac{1}{4\pi a^2} \int d\tau^+ d\tau^- \partial_+ X^\mu \partial_- X_\mu . \qquad (3)$$

In addition there will be some determinants. On the physical grounds discussed above, the result cannot be the usual covariant theory. We can make a good guess as to the difference; we will then verify the self-consistency of this guess by other means. The measure in (1) derives from the physical motion of the underlying gauge and Higgs fields, and so should be built out of physical objects such as the induced metric

$$h_{ab} = \partial_a X^\mu \partial_b X_\mu . \qquad (4)$$

There is no intrinsic metric present in the initial theory, and we will not introduce one. It is then plausible that the determinants are the same as found by Polyakov [6], but built out of the induced metric rather than the intrinsic metric. The Polyakov determinant in conformal gauge is e^{iS_L}, where, in terms of Polyakov's intrinsic metric e^ϕ,

$$S_L=\frac{26-D}{48\pi}\int d\tau^+ d\tau^-\,\partial_+\phi\,\partial_-\phi\,. \qquad (5)$$

Substituting the induced conformal gauge metric h_{+-} for e^ϕ one obtains [9]

$$S_L'=\frac{26-D}{48\pi}\int d\tau^+ d\tau^-\frac{\partial_+^2X\cdot\partial_-X\,\partial_+X\cdot\partial_-^2X}{(\partial_+X\cdot\partial_-X)^2}\,. \qquad (6)$$

The appearance of nonpolynomial terms in (6) may seem alarming. However, we shall see momentarily that these terms are perfectly well defined in an expansion about the long-string vacuum.

We will not carry out the above procedure explicitly. Instead we will use an idea employed by David, Distler, and Kawai (DDK) in a similar situation [10]. That is, we consider a world-sheet field theory with *naive* measure, absorbing the Jacobian into general coefficients in the action. The only restriction on the action is conformal invariance, a remnant of the coordinate invariance of the original action. We differ from DDK in two respects: (i) We have no intrinsic metric and so no Liouville field, and (ii) we *do* allow in the action terms which have an arbitrary, not necessarily polynomial, dependence on $\partial_+X\cdot\partial_-X$ as in Eq. (6). The inclusion of such terms is motivated by the preceding heuristic discussion. Such terms are sensible in an effective theory of long strings because $\partial_+X\cdot\partial_-X$ has a large classical expectation value for long strings. We are describing here an *effective string theory*, valid only for a string whose length is great compared to the fundamental scale a. As with other effective theories, there will be an infinite number of terms, suppressed by powers of a, and the theory breaks down when extrapolated to short strings. It cannot, for example, be expanded around $X=0$ to compute excitations of a small closed string.

To proceed, it is convenient to periodically identify space in the X^1 direction with a very large radius R, and to consider strings that wind once around this direction:

$$X^\mu(\tau,\sigma+2\pi)=X^\mu(\tau,\sigma)+2\pi R\delta_1^\mu\,. \qquad (7)$$

This identification avoids infrared divergent expressions. In the sector with winding number one, we expand around the classical ground state of the leading-order action (3),

$$X_{\rm cl}^\mu=e_+^\mu R\tau^+ + e_-^\mu R\tau^-\,, \qquad (8)$$

where the periodicity, equation of motion, and Virasoro constraints imply that $e_\pm^\mu$ are null and that $e_+\cdot e_-=-\frac{1}{2}$.

Next we write the general Lagrangian in an expansion in powers of R^{-1}, where each *first* derivative of X^μ is of order R. Every term must have world-sheet dimension (1,1). We exclude terms proportional to the leading-order equation of motion $\partial_+\partial_-X^\mu$, which can be removed by field redefinition, and terms proportional to the leading-order constraints $\partial_\pm X\cdot\partial_\pm X$, which vanish weakly between physical states up to higher-order operators. Through order R^{-2} the only possible terms (up to total derivatives) are then

$$S=\frac{1}{4\pi}\int d\tau^+ d\tau^-\left[\frac{1}{a^2}\partial_+X^\mu\partial_-X_\mu+\beta\frac{\partial_+^2X\cdot\partial_-X\,\partial_+X\cdot\partial_-^2X}{(\partial_+X\cdot\partial_-X)^2}+O(R^{-3})\right]. \qquad (9)$$

These are the same terms already considered in Eqs. (3) and (6), except that for now we allow a general coefficient for the induced metric Polyakov term. This action is invariant [i.e., $\delta S\le O(R^{-2})$] under the modified conformal transformation

$$\delta X^\mu=\epsilon^-(\tau^-)\partial_-X^\mu-\frac{\beta a^2}{2}\partial_-^2\epsilon^-(\tau^-)\frac{\partial_+X^\mu}{\partial_+X\cdot\partial_-X}\,, \qquad (10)$$

and similarly for $(+\leftrightarrow-)$. The Noether procedure gives the leading correction to the energy-momentum tensor,

$$T_{--}=-\frac{1}{2a^2}\partial_-X\cdot\partial_-X+\frac{\beta}{2}\frac{\partial_+X\cdot\partial_-^3X}{\partial_+X\cdot\partial_-X}+O(R^{-2})\,, \qquad (11)$$

which obeys $\partial_+T_{--}\le O(R^{-2})$. Note that while the leading modification of S in (9) occurs at order R^{-2}, the resulting modification of T_{--} is order R^{-1} because the first term in the conformal transformation (10) is of order R. This will be seen shortly to have important consequences.

We now expand around the long-string vacuum. In terms of the fluctuation field $Y^\mu=X^\mu-X_{\rm cl}^\mu$ the Lagrangian becomes

$$\mathcal{L}=-\frac{R^2}{8\pi a^2}+\frac{1}{4\pi a^2}\partial_+Y\cdot\partial_-Y+\frac{\beta}{\pi R^2}(\partial_+^2Y\cdot e_-)(e_+\cdot\partial_-^2Y)+O(R^{-3})\,. \qquad (12)$$

To obtain the $T_{--}T_{--}$ operator product to order R^0, we keep terms in T_{--} through order R^{-1},

$$T_{--}=-\frac{R}{a^2}e_-\cdot\partial_-Y-\frac{1}{2a^2}\partial_-Y\cdot\partial_-Y-\frac{\beta}{R}e_+\cdot\partial_-^3Y+O(R^{-2})\,, \qquad (13)$$

and to the necessary order the YY propagator is the unmodified $-a^2\ln(\tau^+\tau^-)$. The operator product is then

$$T_{--}(\tau_-)T_{--}(0)=\frac{D+12\beta}{2(\tau^-)^4}+\frac{2}{(\tau^-)^2}T_{--}(0)+\frac{1}{\tau^-}\partial_-T_{--}(0)+O(R^{-1})\,. \qquad (14)$$

The order-R^0 shift in the central charge, $c=D+12\beta$, arises from the cross term between the R^1 and R^{-1} terms in T_{--}.

This is something new: a D-dimensional Poincaré-invariant conformal field theory consisting of D bosonic fields with variable central charge. In fact, there is a theorem that this is impossible: The right- and left-moving world-sheet currents associated with spacetime translations are separately conserved in a conformal field theory, and so can be written as the gradients of D free fields, with central charge D. Here the proof of separate conservation [11] breaks down because there are operators of negative dimension, inverse powers of $\partial_+X\cdot\partial_-X$.

The condition that the coordinate invariance not be anomalous fixes $c=26$ and so

$$\beta=\beta_c\equiv\frac{D-26}{12}\,. \qquad (15)$$

This is the precise value obtained earlier in Eq. (6) by naive reasoning—unlike the case studied by DDK there is not even a finite renormalization. We conclude that the long-distance theory of Nielsen-Olesen and QCD strings is given by the action (9) with coefficient β_c.

As a further check, we now consider the spectrum. Expanding

$$\partial_-Y^\mu=a\sum_{m=-\infty}^{\infty}\alpha_m^\mu e^{-im\tau_-} \qquad (16)$$

gives the Virasoro generators

$$L_n=\frac{R}{a}e_-\cdot\alpha_n+\frac{1}{2}\sum_{m=-\infty}^{\infty}:\alpha_{n-m}\cdot\alpha_m:+\frac{\beta_c}{2}\delta_{n,0}-\frac{\beta_c a n^2}{R}e_+\cdot\alpha_n+O(R^{-2})\,. \qquad (17)$$

The operator product (14) implies that the generators satisfy a Virasoro algebra,

$$[L_m,L_n]=(m-n)L_{m+n}+26(m^3-m)\delta_{m,0}/12\,;$$

this algebra can be used to determine the normal-ordering constant in L_0. There is a second copy of the Virasoro algebra from the left movers.

The quantum ground state $|k,k;0\rangle$ is an eigenvector of α_0^μ and $\tilde\alpha_0^\mu$ with common eigenvalue ak^μ, and is annihilated by the lowering operators. The total momentum of the string is

$$p^\mu=\frac{R}{2a^2}(e_+^\mu+e_-^\mu)+\frac{1}{2a}(\alpha_0^\mu+\tilde\alpha_0^\mu)\,, \qquad (18)$$

the first term coming from $X_{\rm cl}^\mu$ and the second from the fluctuation Y^μ. Imposing the physical state condition [12] $L_0=\tilde L_0=1$ gives $k^1=0$, and for the total rest energy of the string

$$(-p^2)^{1/2}=\frac{R}{2a^2}-\frac{D-2}{12R}+O(R^{-3})\,. \qquad (19)$$

The first term is the potential energy for a string of length $2\pi R$, and the second is a Casimir energy. The value of the Casimir energy is an important check: On general grounds one expects an energy $-1/12R$ per physical degree of oscillation, which is precisely our result. This is rather nontrivial in light of a general argument [13] that the Casimir energy should be $-2/R$ for matter central charge 26: Evidently the general argument is inapplicable because the ground-state vertex operator is dressed with a power of $\partial_+X\cdot\partial_-X$.

As another check consider the first excited states with one right mover, $E\cdot\alpha_{-1}|k,\tilde k;0\rangle$. The L_1 condition is $E\cdot v=0$, where $v^\mu=Re_-^\mu/a+ak^\mu+\beta_c ae_+^\mu/R$. This has $D-1$ solutions, but v^μ is null as a result of the $L_0=\tilde L_0=1$ condition, so the solution $E^\mu=v^\mu$ is null, leaving $D-2$ physical oscillations. Actually, this result and the generalization to all mass levels follows immediately once the central charge of the Virasoro algebra is known to be 26.

Presumably this procedure can be continued to find higher-order corrections to the action, stress tensor, and spectrum. The first corrections to the action (9) are of order R^{-4} and are of three types: (i) order-$a^2\beta^2$ terms required for classical conformal invariance; (ii) order-$a^2\beta$ terms required for quantum conformal invariance; (iii) terms independent of β but proportional to new free parameters. One new parameter, of order a^0, is the coefficient in the original action (2) of the extrinsic curvature term, the first classical correction to the Nambu-Goto action [14]. This term is therefore less important for long strings than the $O(R^{-2})$ effect from the measure.

It is interesting to make contact with the Liouville theory. If we introduce an intrinsic metric on the world sheet, we should obtain a conformal field theory with Liouville field ϕ. How might this differ from that obtained by DDK? Consider in the ordinary Liouville theory the operator

$$\mathcal{L}'=-\lambda(\partial_+X_\mu\partial_+X_\nu-D^{-1}\eta_{\mu\nu}\partial_+X\cdot\partial_+X)$$
$$\times(\partial_-X^\mu\partial_-X^\nu-D^{-1}\eta^{\mu\nu}\partial_-X\cdot\partial_-X)e^{\gamma\phi}\,, \qquad (20)$$

with $(12)^{1/2}\gamma=-(49-D)^{1/2}+(25-D)^{1/2}<0$. This is a (1,1) tensor and thus can be consistently added to the Liouville Lagrangian. Why is it not usually considered? It has four derivatives and so is nonrenormalizable, but in an effective theory such as we are considering this would not exclude it. Also, it couples the Liouville field to the X^μ. One might have hoped that these would remain decoupled, but we see no physical reason to expect this. Adding $\mathcal{L}'$ to the action produces in the long string the effective Liouville potential $\lambda R^4 e^{\gamma\phi}/4$, which grows in the direction of negative ϕ. Combined with the usual Liouville potential which grows with positive ϕ, the Louiville field is massive and drops out of the low-energy theory. Of course $\mathcal{L}'$ is one of an infinite number of new terms which can appear, but the conclusions remain the same. Integrating out ϕ leaves a theory with only X^μ, which is the same theory that we have arrived at above [15].

For the hope that QCD can be exactly reformulated as a string theory, our results are useful but not necessarily encouraging. We now understand the low-energy limit of the world-sheet field theory. But this low-energy limit could result from one of many different short-distance field theories, or from no two-dimensional field theory at all. If the short-distance dynamics is described by a two-dimensional field theory, this would be from the spacetime point of view a string theory. It may be productive to take the point of view of a two-dimensional physicist, who has found the analog of pion theory, a nonrenormalizable low-energy theory with the correct symmetries and degrees of freedom, and who is trying to find the right short-distance theory.

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[7] We use Minkowski light-cone world-sheet coordinates $\tau^\pm = \tau \pm \sigma$. The spacetime metric is $- + + \cdots$

[8] One way to do this is as follows. First, take conformal-gauge coordinates $h_{++} = h_{--} = 0$ on the world sheet: The X integrations are then constrained by $\delta(h_{++})\delta(h_{--})$. Write *one* delta function, say for h_{--}, in an integral representation with Lagrange multiplier λ^{--}. Now make a world-sheet coordinate transformation to variables X' so as to remove λ^{--} from the action, which then takes the form (3) in terms of X'. The final delta function is removed by integration over λ^{--} at fixed X'. There is some literature on the path-integral treatment of the Nambu-Goto theory of Eqs. (1) and (2), for example, J.-L. Gervais and B. Sakita, Phys. Rev. Lett. **30**, 716 (1973); E. S. Fradkin and A. A. Tseytlin, Ann. Phys. (N.Y.) **143**, 413 (1982); M. Lüscher, K. Symanzik, and P. Weisz, Nucl. Phys. **B173**, 365 (1980); J. Govaerts, Int. J. Mod. Phys. A **4**, 173 (1989); T. R. Morris, Nucl. Phys. **B341**, 443 (1990), and there is some overlap between these papers and our results. In fact, the paradox which we solve was already noted in the 1973 paper of Gervais and Sakita. However, our main result, the conformal field theory (9) with central charge not equal to D, and the resulting covariant quantization with $D-2$ oscillators, appears to be new.

[9] The measure is local, so the Jacobians should be as well. We therefore guess that the Polyakov determinant will appear in its local conformal-gauge form rather than its gauge-invariant nonlocal form.

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