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Universal Noninteger "Ground-State Degeneracy" in Critical Quantum Systems

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One-dimensional critical quantum systems have a universal, intensive "ground-state degeneracy," g, which depends on the universality class of the boundary conditions, and is in general noninteger. This is calculated, using the conjectured boundary conditions corresponding to a multichannel Kondo impurity, and shown to agree with Bethe-ansatz results. g is argued to decrease under renormalization from a less stable to a more stable critical point and plays a role in boundary critical phenomena quite analogous to that played by c, the conformal anomaly, in the bulk case.

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The logarithm of the partition function $\ln Z$ for a one-dimensional quantum critical system is well known [1] to contain a universal term $\ln Z = (\pi/6v)(l/\beta)c$, where v is the velocity of "light," l is the length, $\beta = 1/T$ the inverse temperature, and c is the conformal anomaly parameter which depends on the universality class. This formula is valid when $l \gg v\beta$. There is one other term which could appear in $\ln Z$, in the large l limit, without violating scale invariance, namely, a constant, length-independent zero-temperature entropy, $S(0) = \ln g$, where g is the "ground-state degeneracy." Since this term does not depend on length, it is natural to expect that it is associated with the boundaries of the system, i.e., $g = g_1 g_2$, where the two factors arise from the two boundaries. For periodic boundary conditions g must be 1 since there effectively is no boundary, but that need not be true in general. For any finite l the ground-state degeneracy must be an integer, since the spectrum is discrete. However, in the $l \rightarrow \infty$ limit, when the spectrum is continuous, g can be noninteger. Indeed, in this limit, g is determined by the asymptotic spectrum at large quantum number n, where the energy levels take the form $E = vn\pi/l$ (n integer). The asymptotic degeneracy takes the form

$$D(n) \rightarrow (g/2)(c/6n^3)^{1/4} \exp(2\pi\sqrt{cn/6})$$

and there is no *a priori* reason for g to be integer. Standard finite-size-scaling hypotheses imply that all corrections to $S(T)$ (both extensive and intensive), coming from irrelevant operators, vanish with higher powers of T, so that c and g should both be universal. (We gave a brief discussion of finite-size scaling in the presence of boundaries elsewhere [2].) Branches of excitations separated by a gap, Δ, from the ground state should make exponentially small contributions to S of $O(\exp(-\Delta/T))$.

As Cardy has emphasized [3], a conformally invariant boundary condition must correspond to a boundary *state* since we can interchange the role of space and imaginary time (see Fig. 1), and so the partition function becomes $Z_{AB} = \langle A|\exp[-(l/v)H_P]|B\rangle$. Here H_P is the Hamiltonian with periodic boundary conditions (on an interval of length βv) and $|A\rangle, |B\rangle$ are the boundary states. Now taking the limit $l \gg \beta v$, this becomes $Z_{AB} \rightarrow \langle A|0\rangle \times \exp[-(l/v)E_0]\langle 0|B\rangle$, where $|0\rangle$ is the ground state of H_P and E_0 is the ground-state energy. Thus the degen-

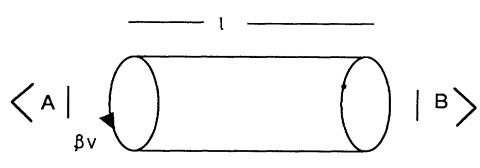

FIG. 1. The space-time is periodic in one direction and has boundary conditions or boundary states at the ends.

eracy (for each boundary) is $g_A = \langle 0|A\rangle$. (The phase of $|0\rangle$ can be chosen such that $\langle 0|A\rangle$ is real and positive for all boundary states, $|A\rangle$.)

A third way of looking at g occurs in a two-dimensional classical statistical critical system defined on a long strip (of length l and width β) with nontrivial boundary conditions at the ends of the strip. The free energy produced by the boundary is of the form $\beta f_B = -\ln Z_B = \beta f_0 - \ln g$, where the free energy per unit length f_0 is not universal but the length-independent term $\ln g$ is.

g can be conveniently calculated from the integers, n^a_{AB}, specifying the low-temperature partition function for given boundary conditions $|A\rangle$ and $|B\rangle$:

$$Z_{AB} = \mathrm{Tr}\exp(-\beta H_{AB}) = \sum_a n^a_{AB}\chi_a[\exp(-v\pi\beta/l)]\,.$$

Here χ_a is the character of the ath conformal tower:

$$\chi_a[\exp(-v\pi\beta/l)] = e^{v\pi\beta c/24l}\,\mathrm{tr}_a e^{-(v\pi\beta/l)L_0}\,,$$

where L_0 is a generator of the Virasoro algebra.

The limit $\beta v/l \to 0$ can be conveniently expressed in terms of the matrix S^b_a representing a modular transformation on the characters:

$$\chi_a[\exp(-v\pi\beta/l)] = \sum_b S^b_a\chi_b[\exp(-4\pi l/v\beta)]\,. \tag{1}$$

Taking the limit $\beta v/l \to 0$ on the right-hand side of Eq. (1), only the ground state contributes, so

$$Z_{AB} \to \sum_a n^a_{AB}S^0_a\chi_0[\exp(-4\pi l/v\beta)]$$
$$\to \exp(\pi lc/6v\beta)\sum_a n^a_{AB}S^0_a\,.$$

Thus we see that

$$g_Ag_B = \sum_a n^a_{AB}S^0_a\,.$$

The consistency of this formula with $g_A = \langle 0|A\rangle$ puts certain constraints on the possible boundary states $|A\rangle$ and on the parameters n^a, as argued by Cardy.

As a first simple example, we consider the two-dimensional Ising model at its critical point. There are two conformally invariant boundary conditions, free or fixed. Cardy has calculated the corresponding boundary states and these give $g=1$ and $1/\sqrt{2}$, respectively.

Our most important example is the Kondo effect, which may be formulated as a problem in (1+1) space-time dimensions. The Hamiltonian density is

$$\mathcal{H} = iv\psi_L^{i\alpha\dagger}\frac{d\psi_{i\alpha L}}{dx} - iv\psi_R^{i\alpha\dagger}\frac{d\psi_{i\alpha R}}{dx}$$
$$+ (\pi/2)\lambda_K\delta(x)\mathbf{S}\cdot(\psi_L^{i\alpha\dagger}+\psi_R^{i\alpha\dagger})\tfrac{1}{2}\boldsymbol{\sigma}^\beta_\alpha(\psi_{i\beta L}+\psi_{i\beta R})\,.$$

Here ψ_L and ψ_R are left- and right-moving fermions with spin index $\alpha=1,2$ and "flavor" index $i=1,2,\ldots,k$. $\mathbf{S}$ is a quantum spin operator of size s.

We treated this problem [2,4] by decomposing the fermions into charge, spin, and flavor degrees of freedom, using a conformal embedding involving a free boson representing charge, a level k, SU(2) Wess-Zumino-Witten (WZW) field representing spin, and a level 2, SU(k) field representing flavor. We then hypothesized that, under renormalization to the low-temperature fixed point, the impurity spin disappears, leaving behind only a modified, conformally invariant, boundary condition on the charge, spin, and flavor fields. Only certain special choices of the multiplicities, n^a, or equivalently of the boundary conditions, correspond, in a simple way, to free fermions. These choices force the two nontrivial WZW theories to conspire together to produce trivial behavior. Other choices of the n^a's lead to nontrivial behavior, producing, for example, fractional energy-level spacings. These presumably correspond to "boundary conditions" which are not local in the fermion basis. We conjectured that the low-temperature fixed point in the Kondo problem corresponds to the set of integers n^a obtained from those corresponding to trivial boundary conditions by applying the fusion rules to add the impurity spin to the conduction-electron conformal towers, corresponding to the physical picture of the conduction electrons screening, or adsorbing, the impurity. The corresponding spectrum was in excellent agreement with that obtained from Wilson's numerical renormalization-group approach in the case $k=2$, $s=\frac{1}{2}$.

Let us first calculate g for the free-fermion case ($\lambda_K=0$) and vanishing boundary conditions, at both ends as used in Refs. [2] and [4]. The partition function is written

$$Z = \sum_{Q,j,\rho} n_{\mathrm{FF}}^{Q,j,\rho}\chi_{\mathrm{U(1)},Q}\chi_{\mathrm{SU(2)},j}\chi_{\mathrm{SU}(k),\rho}\,.$$

Here $\chi_{\mathrm{SU(2)},j}$ and $\chi_{\mathrm{SU}(k),\rho}$ are the characters of SU(2), spin j and SU(k), representation ρ, respectively. $\chi_{\mathrm{U(1)},Q}$ is the sum of characters of U(1) over all charges equal to Q, mod$2k$, i.e.,

$$\chi_{\mathrm{U(1)},Q} = \exp(\pi v\beta/24l)\sum_{n=-\infty}^{\infty}\exp[-(v\pi\beta/l)(Q+2kn)^2/4k]\prod_{m=1}^{\infty}[1-\exp(-v\pi\beta m/l)]^{-1}\,.$$

The integers $n_{\mathrm{FF}}^{Q,j,\rho}$ are the free-fermion multiplicities found by Altschüler, Bauer, and Itzykson [5] (ABI). In the limit $l \gg v\beta$, we find

$$Z \to \exp(\pi lc/6v\beta)(1/\sqrt{2k})\sum_{Q,j,\rho} n_{\mathrm{FF}}^{Q,j,\rho}S^0_{j,\mathrm{SU(2)}}S^0_{\rho,\mathrm{SU}(k)}\,.$$

ABI show that any SU(k) representation ρ which appears together with the SU(2) representation j is obtained from j by a bijection such that

$$S^0_{\rho(j),\mathrm{SU}(k)} = \sqrt{2/k}\,S^0_{j,\mathrm{SU(2)}}\,.$$

Furthermore, each SU(2) representation appears k times [once with each of the SU(k) representations obtained from the bijection and the action of the group center, Z_k]. Thus we find the degeneracy for free-fermion boundary conditions, g_F, is given by

$$g_F^2=\sum_{j=0}^{k/2} S_j^0 S_j^0 .$$

The matrix $S_{j'}^j$ is both unitary and real symmetric [6]:

$$S_{j'}^j=\sqrt{2/(2+k)}\sin[\pi(2j+1)(2j'+1)/(2+k)] .$$

Hence, $g_F=1$.

To calculate g at the Kondo fixed point, we simply modify [2,4] the integers $n^{Q,j,\rho}$ according to the fusion rules, i.e.,

$$n_{\mathrm{KF}}^{Q,j,\rho}=\sum_{j'} N_{j's}^j n_{\mathrm{FF}}^{Q,j',\rho} ,$$

where $N_{j's}^j$ is the fusion rule coefficient [7] for the SU(2)-level k theory, giving the number of distinct ways that the representation j occurs in the operator product expansion of two fields transforming according to the representations j' and s, respectively. (s is the spin of the impurity.) We now have "Kondo" boundary conditions at one end and free fermion ones at the other. Hence we find

$$\begin{aligned} g_K &= (1/\sqrt{2k})\sum_{Q,j,\rho} n_{\mathrm{KF}}^{Q,j,\rho} S_{j,\mathrm{SU}(2)}^0 S_{\rho,\mathrm{SU}(k)}^0 \\ &= (1/\sqrt{2k})\sum_{Q,j,\rho,j'} N_{j's}^j n_{\mathrm{FF}}^{Q,j',\rho} S_{j,\mathrm{SU}(2)}^0 S_{\rho,\mathrm{SU}(k)}^0 . \end{aligned}$$

The sum over j can be done using the Verlinde formula [8]:

$$\sum_j N_{j',s}^j S_j^0 = S_{j'}^0 S_s^0 / S_0^0 .$$

The remaining sum can then be done as is in the free case, leaving

$$g_K=\frac{S_s^0}{S_0^0}=\frac{\sin[\pi(2s+1)/(2+k)]}{\sin[\pi/(2+k)]} .$$

This formula agrees exactly with the degeneracy obtained from the Bethe-ansatz solution of the Kondo problem [9]. It thus provides direct confirmation of the spectrum (i.e., the boundary state) conjectured in Ref. [2] for all values of the impurity spin s and the number of flavors k. $U_xY_{1-x}Pd_3$ has been proposed as a realization of the two-channel, $s=\frac{1}{2}$ Kondo effect. Experimental measurements [10] are suggestive of a zero-temperature entropy of $\frac{1}{2}\ln 2$, as predicted by the above general formula.

We note that the Verlinde formula can also be used to calculate arbitrary matrix elements of the Kondo state. Cardy established the general formula

$$\sum_b S_b^a n_{AB}^b = \langle A|a;0\rangle\langle a;0|B\rangle , \tag{2}$$

where a,b label conformal towers in a general field theory and the state $|a;0\rangle$ is a direct product of left and right copies of the highest weight state in the ath tower. The same procedure as above leads to the result

$$\langle Q,j,\rho;0|K\rangle/\langle Q,j,\rho;0|F\rangle = S_s^j/S_0^j .$$

This formula will be very useful in calculating the correlation functions for the Kondo problem [11]. [Actually, Cardy only proved Eq. (2) assuming nondegenerate characters. However, his proof can be easily extended to this case, where the only degeneracies are those that arise from charge-conjugation symmetry.]

In all examples which we have studied, the degeneracy g *decreases* upon flowing between repulsive and attractive fixed points. For instance, in the Ising model a free boundary condition represents an unstable fixed point upon the application of a magnetic field at the boundary which drives the system to the fixed boundary condition. g decreases from 1 to $1/\sqrt{2}$. In the Kondo effect, the degeneracy at the unstable zero-coupling fixed point is simply that of the decoupled spin, namely, $2s+1$. At the stable strong-coupling fixed point this is always *reduced* to

$$\sin[\pi(2s+1)/(2+k)]/\sin[\pi/(2+k)] .$$

The multichannel fixed points in the Kondo problem are unstable against channel asymmetries in the Kondo coupling. For instance, in the two-channel case, with $s=\frac{1}{2}$, if one channel couples more strongly to the impurity than the other then it is believed that the system crosses over to a fixed point at which one of the channels screens the impurity (undergoing a conventional single-channel Kondo effect) and the other channel is unaffected [12]. This corresponds to g decreasing from $\sqrt{2}$ to 1. We also note that g appears always to decrease upon taking the infinite-length limit, compared to its finite-length (integer) value (i.e., if we take $T\to 0$ for finite l, we always obtain an integer since the spectrum is discrete, whereas if we take $l\to\infty$ *before* taking $T\to 0$, we appear to always obtain a smaller value of g, in some cases noninteger). At the (symmetric) fixed point, the finite-length ground state is the highest weight state in the conformal tower with spin s, charge 0, and singlet flavor representation. This is a spin-s multiplet of degeneracy $2s+1$. Again this is reduced at $l\to\infty$.

We conjecture that g always decreases under renormalization from a less stable to a more stable critical point in the same *bulk* universality class. A similar decrease of the conformal anomaly, c, occurs when the bulk universality class changes. A naive argument for this decrease is that, in both cases, as we go to lower-energy scales states that appeared to be gapless may exhibit small gaps, hence reducing c and g. A perturbative argument for this "g theorem" can be constructed in a very similar way to the one given in the case of c in Ref. [13]. We consider a barely relevant boundary interaction in the Euclidean space action, $-\lambda\int d\tau\,\phi(\tau)$, where the operator ϕ has dimension $1-y$ with $0<y\ll 1$. ϕ has a unit-normalized two-point function, $\langle\phi(\tau_1)\phi(\tau_2)\rangle=|\tau_1-\tau_2|^{-2(1-y)}$, and

a three-point function,

$$\langle\phi(\tau_1)\phi(\tau_2)\phi(\tau_3)\rangle = -b|(\tau_1-\tau_2)(\tau_2-\tau_3)(\tau_3-\tau_1)|^{-(1-y)},$$

where b is real. The β function is $\beta = y\lambda - b\lambda^2$, with a fixed point at $\lambda = y/b \ll 1$. We calculate $\ln Z$ perturbatively, expressing the result in terms of the renormalized coupling constant, $\lambda(l)$, giving

$$\ln g(\lambda(l)) = \ln g(0) - \lambda(l)^2 y\pi^2 + \tfrac{2}{3}\lambda(l)^3\pi^2 b + O(\lambda^4).$$

Evaluating g at the fixed point gives $\delta g/g = -\frac{1}{3}\pi^2 y^3/b^2 < 0$.

It seems that it should be possible to construct a nonperturbative proof of the g theorem similar to the proof of Zamolodchikov for the c theorem [14]. However, so far, we have been unable to find a satisfactory proof.

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