Instability of Taylor-Couette Flow of Helium II

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We report observations of the instability of Taylor-Couette flow of helium II. The results are in substantial agreement with recent calculations of Barenghi and Jones using the full modern equations of the flow of helium II. This result is the first instance of success of a linear-instability theory with the equations of helium II and is the strongest evidence available that the equations of motion and the boundary conditions used in the analysis are correct.

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The study of liquid-helium Taylor-Couette flow (flow of a fluid between rotating cylinders) began in 1957 with the stability analysis of Chandrasekhar and Donnelly [1]. They recognized the importance of the problem as a rigorous test of the two-fluid equations of motion [4]. The equations of motion used by Chandrasekhar and Donnelly were not complete since the vortex tension term was not included. Today the accepted equations are known as HVBK after Hall, Vinen, Bekarevich, and Khalatnikov [2].

Recently Donnelly and LaMar reviewed all existing experimental and theoretical work on Taylor-Couette flow of helium II [3]. They noted first of all that the experimental data suggest that instability in helium II occurs when the inner or outer cylinder is rotating.

During preparation of the review, Barenghi and Jones [4] of the University of Newcastle upon Tyne decided to undertake a fresh look at the stability theory using the modern HVBK equations of motion. These equations contain both mutual friction and vortex tension terms [4]. Barenghi and Jones found that for the rotation of the outer cylinder the superfluid is unstable to *nonaxisymmetric* perturbations as the wave number $\mathbf{k}$ approaches 0. This is in strong contrast to the classical result where the fluid is stable to any rotation of the outer cylinder. Since the instability sets in as $k \rightarrow 0$, the experimental results should depend greatly on the aspect ratio and end effects. The situation is quite different when only the inner cylinder rotates. The first instability in the vortex array is temperature dependent. As one approaches the lambda point the critical Reynolds number approaches the classical value from above. As the temperature is reduced the critical Reynolds number increases and the critical wave number decreases. Eventually, the critical wave number again becomes zero causing end effects to predominate.

In addition to temperature dependence, Barenghi and Jones found that the critical Reynolds number depends strongly on the radius ratio of the cylinders. As the radius ratio approaches 1 both the critical Reynolds number and the wave number of the instability increase. To facilitate comparison of results to theory we report on measurements conducted at a radius ratio near unity (a very narrow annulus) and at a variety of temperatures. Furthermore, we concentrate exclusively on inner cylinder rotation.

There is a profound difference between the flow of helium II in an annulus and that for a classical fluid. For a classical fluid the base state is that of laminar flow $v = Ar + B/r$ no matter how slow one rotates. Here A and B are functions of the angular velocities and the radii of the cylinders. For helium II, slow rotation corresponds to normal-fluid flow of the form

$$\mathbf{v}_n = (Ar + B/r)\hat{\boldsymbol{\phi}} \tag{1}$$

as in the classical case, but the superfluid flow has the form

$$\mathbf{v}_s = \frac{m\kappa}{2\pi R_1}\hat{\boldsymbol{\phi}}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{2}$$

where R_1 is the radius of the inner cylinder, $\kappa = h/m$, h is Planck's constant, and m is the mass of the helium atom. At higher rates of rotation, quantized vortices enter the annulus, and the superfluid flow becomes more complex. It is this state of normal-fluid motion and vortex-filled superfluid that is the base state considered in the stability theory and which must act as a continuum state for the theory to be valid. The number of vortices per unit area of helium II, n, in the annulus is roughly given by the ratio of the vorticity to the quantum of circulation [2],

$$n = \frac{|\mathrm{curl}\,\mathbf{v}_s|}{\kappa} = \frac{2A}{\kappa}. \tag{3}$$

We find experimentally that (3) is obeyed above some threshold value of A, so the condition that a continuum of vorticity exists for a well-defined base state is satisfied.

Since we are primarily concerned with the rotation of the inner cylinder, we nondimensionalize the inner cylinder angular velocity Ω_1 with the Reynolds number defined in the standard way as

$$\mathrm{Re}_1 = \Omega_1 R_1 d/\nu_n, \tag{4}$$

where d is the gap size and the kinematic viscosity ν_n is defined as the ratio of dynamic viscosity to normal-fluid density. Other parameters describing the system are the radius ratio $\eta = R_1/R_2$ and the aspect ratio $\Gamma = h/d$.

Our apparatus consists of two concentric cylinders of radii 1.9508 and 1.9982 cm. A height of $h = 9.436$ cm separates the end caps which rotate with the inner cylinder. Thus we have $\eta = 0.97628$ and $\Gamma = 199$. We have chosen these parameters to allow us to compare our data with the theory of Barenghi and Jones.

We used second-sound resonances within the cavity to detect the quantized vortices. The second sound is emitted by an ac voltage across a resistive heater. The heater is a 20-μm-thick Evenohm wire attached lengthwise along the inner cylinder. The resulting temperature waves propagate axially, azimuthally, and radially at twice the input frequency. The second sound is detected by a painted strip of Aquadag carbon compound 5 mm wide and approximately 5 μm thick. This detector is also aligned lengthwise along the inner cylinder separated from the heater by 90°. It carries a small dc bias current dissipating between 1 and 2 mW of power into the working fluid. The temperature fluctuations from the heater cause corresponding resistance fluctuations which are detected as a weak ac signal. The signal is amplified and then transferred to the laboratory frame through slip rings to a phase-sensitive lock-in amplifier.

The second-sound waves are attenuated primarily by losses at the walls, and by the presence of quantized vortex lines. The excess attenuation α due to the vortices is measured by comparing the resonance amplitude of the rotating system with that of the stationary system. Specifically,

$$\alpha = \frac{\pi\Delta}{u_2}\left(\frac{V}{V_0} - 1\right), \tag{5}$$

where V_0 is the resonance amplitude without rotation and V is the amplitude with rotation [2]. Δ is the full width at half the maximum power of the resonance and u_2 is the second-sound velocity. We were able to achieve a resolution in $V/V_0 - 1$ of 5×10^{-4} which translates into a detection of vortex core material present in a concentration of 1 part in 10^{14}.

The primary resonance mode used was the fundamental axial mode. The presence of an axial mode in an axially symmetric geometry was somewhat mysterious. However, small gaps between the end caps and the outer cylinder were left at the ends of the cylinders to allow relative rotation. We believe that the second sound was passing through these gaps and a resonant mode was generated with a wavelength longer than twice the measured height. This was verified by a calculation of the wavelength from frequency and second-sound velocity data. No other higher axial modes were detected.

The vortex array in laminar flow between infinite cylinders is parallel to the axis of rotation. Since second sound is attenuated only when it propagates perpendicular to the vortices, and axial mode should not detect vortex lines. However, in a finite geometry the end caps of the cylinders do not rotate with the laminar profile, creating vorticity with a component perpendicular to the axis of rotation. Using the axial mode we were able to measure this vorticity and found it to be 2 or 3 times weaker than the bulk vorticity as detected by the azimuthal resonance.

We expect the transition to be marked by some deformation in the vortex array configuration since the normal fluid is changing its flow pattern. Such a deformation would presumably be evident to both axial and azimuthal second-second measurements. This indeed was the case. In fact, the extra attenuation due to the deformation was nearly the same for both resonances. However, the percentage change in attenuation due to the instability of the array was greater for the axial resonant mode than for the azimuthal. This results from the larger amount of background attenuation in the azimuthal mode from the many axially oriented lines. Consequently, by using the fundamental axial mode we were able to determine the critical Reynolds number more precisely.

The transition was revealed by a break in the slope of attenuation as a function of Reynolds number. The break in slope was often accompanied by a discontinuity in the attenuation itself. Figure 1 shows data taken at 2.1 K. The squares represent attenuation as Ω_1 was being stepwise increased, and the triangles represent attenuation as Ω_1 was being stepwise decreased. One can see that the transition was free of hysteresis. In fact, no hysteresis was detected for a range of rotational accelerations spanning 3 orders of magnitude. By contrast, in classical Taylor-Couette flow, there is a criterion set forth by Park, Crawford, and Donnelly for passing through the Taylor transition without hysteresis [5]. Namely, only when

$$a^* \equiv \frac{R_1 d^2 L}{\nu^2}\frac{d\Omega_1}{dt} < 6 \tag{6}$$

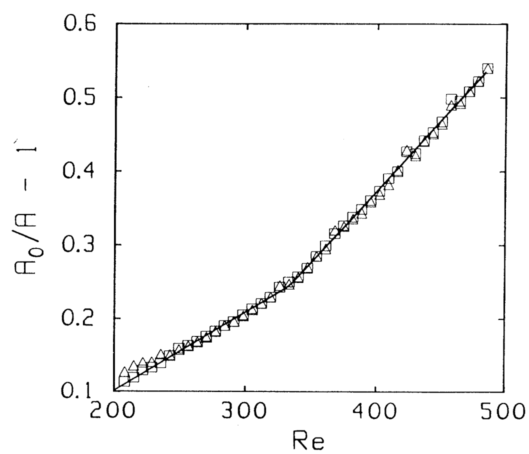

FIG. 1. Second-sound attenuation as a function of inner cylinder Reynolds number at $T = 2.10$ K. The squares and triangles correspond to data taken as the Reynolds number was being stepped up and stepped down, respectively. The line is a least-squares fit to the data.

will the hysteresis be negligible. Because of the very small kinematic viscosity of helium our dimensionless accelerations were in all cases at least 2 orders of magnitude greater than the above criterion. The lack of hysteresis may provide an important clue in understanding the dynamics of the superfluid transition.

The critical Reynolds number was determined by a least-squares fit of the data in Fig. 1, letting Re_c be one of the parameters determined by the fit. The fitting function consisted of two linear regions joined smoothly by a step function,

$$y(\mathrm{Re}) = (a\,\mathrm{Re}+b)f(\mathrm{Re}) + (c\,\mathrm{Re}+d)[1-f(\mathrm{Re})]\,,$$

where

$$f(\mathrm{Re}) = \frac{1}{1+e^{(\mathrm{Re}-\mathrm{Re}_c)/\delta}}\,. \tag{7}$$

For $\mathrm{Re} < \mathrm{Re}_c$, f is close to 1, and for $\mathrm{Re} > \mathrm{Re}_c$, f is close to 0. The critical Reynolds number Re_c and the parameters a, b, c, and d were free parameters while the sharpness δ of the step function was fixed at 5.

Some precautions in fitting were taken to assure an accurate assessment of Re_c. First, we found the value of Re_c to be relatively insensitive to variations of δ between 3 and 8. For δ's beyond this range the fitting routine would not converge. Second, only the data within a certain range of Re were appropriate to use in the fit. For instance, for Re around 2 to 3 times Re_c there was a transition to another flow regime causing a further break in slope. Other times the attenuation strayed from linear growth before or after Re_c. In a majority of cases the choice of which data to fit was straightforward and clear. When the choice was less evident, a larger uncertainty was assigned to Re_c at that temperature.

The critical Reynolds number as a function of temperature is plotted in Fig. 2. The error bars on the graph represent 90%-confidence intervals calculated by the fitting routine except for the cases mentioned above. The open squares are our experimental data points and the solid triangles are calculations from the stability analysis of Barenghi and Jones [6]. The agreement is good for those temperatures where experiment and theory overlap. It appears that there may be a small systematic deviation which could be accounted for by imprecise knowledge of the system dimensions. Since the critical Reynolds number depends strongly on radius ratio, an error of 0.2% in radius ratio would correspond to a shift in Re_c of 20.

The most noteworthy feature of Fig. 2 is the downward turn of the data as $T \to T_\lambda$. The rightmost triangle is the calculated critical Reynolds number at T_λ where helium becomes a purely classical fluid. The approach of the experimental critical value to the classical one shows that we recover the classical result, and the transition we have found is indeed the well-known Taylor transition.

Below 2.1 K the stability analysis finds that the wave number $k = 2\pi d/\lambda$ goes to zero, making quantitative comparisons to experiment difficult. Qualitatively, however, the theory expects a sharp drop of Re_c with decreasing temperature. This is observed experimentally. We found it difficult to determine the critical point as the temperatures dropped below 2.0 K owing to increased rounding of the data. Furthermore, the range of Reynolds numbers usable for fitting became smaller, making the choice of which data to fit increasingly difficult. There often was an additional break in the slope at a Reynolds number not much higher than critical. For instance at 1.975 K we found two breaks in the slope both of which we include in Fig. 2 for completeness. Also the determination of which break corresponded to the onset of quantized vorticity and which corresponded to the Taylor transition became difficult. A separate study of the onset provided some helpful clues, but we nevertheless consider the lower-temperature data more tentative than that which was taken above 2.0 K.

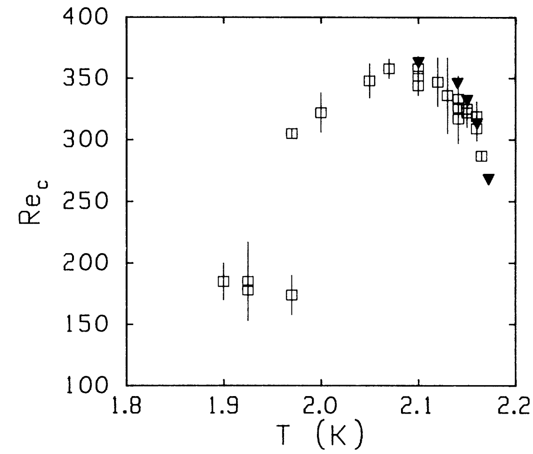

FIG. 2. Temperature dependence of the critical Reynolds number. The squares are experimental data points with errors from the fitting routine. The solid triangles are calculated values from the stability analysis of Barenghi and Jones. The lowest triangle is the critical Reynolds number for He I at 2.172 K.

We have shown the success of the modern equations of motion and the boundary conditions by verifying the linear-stability theory of Barenghi and Jones [4]. Equally important, a new field of study has been opened in He II fluid dynamics. It has been made possible by the invaluable direction provided by the theory, for we would not likely have found this subtle transition without theoretical guidance. Further theoretical work, both linear and nonlinear, and further experimental investigations are called for. Local probes of the vorticity could provide information about the vortex structure. Variations of the aspect ratio could be helpful in uncovering the low-temperature behavior, and an examination of flow states at higher Reynolds numbers where turbulence sets in would also be of interest. As in classical Taylor-Couette flow, many variations on the basic experiment reported here suggest themselves. Adding axial counterflow with a heater is an obvious next step.

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