Nonlinear Nature of Gravitation and Gravitational-Wave Experiments

Demetrios Christodoulou
Courant Institute of Mathematical Sciences, New York University, New York, New York 10012
(Received 17 December 1990; revised manuscript received 20 June 1991)

It is shown that gravitational waves from astronomical sources have a nonlinear effect on laser interferometer detectors on Earth, an effect which has hitherto been neglected, but which is of the same order of magnitude as the linear effects. The signature of the nonlinear effect is a permanent displacement of test masses after the passage of a wave train.

PACS numbers: 04.30.+x, 04.80.+z

The need of taking full account of the nonlinearity of Einstein's equations when one wants to study the generation of gravitational waves from strong sources is generally recognized. However, since the sources are at enormous distances from the Earth, the amplitude of the waves when they reach the detector is so small that it has always been assumed that when treating the waves in the Earth's neighborhood the linearized theory suffices. It is the purpose of this Letter to show that this assumption is in error.

The nonlinearity of Einstein's equations manifests itself in a permanent displacement of the test masses of a laser interferometer detector after the passage of a wave train. Such a permanent displacement, called the "memory" of the gravitational-wave burst [1,2], has long been known to occur [3] within the framework of the linearized theory as a result of an overall change of the second time derivative of the source's quadrupole moment or equivalently of an overall change of the linear momenta of the constituent bodies. As this was the only known cause of a memory effect, it was thought that typical sources, i.e., the coalescense of a neutron star binary, in which little linear momentum is radiated away, will produce bursts with negligible memory. However, we show in this Letter that every burst has a nonlinear memory, due to the cumulative contribution of the effective stress of the gravitational waves themselves. Moreover, for a binary coalescense, the nonlinear memory is of the same order of magnitude as the maximal amplitude of the dynamical part of the burst.

Our treatment is based on the rigorous analysis of the asymptotic behavior of the gravitational field given in [4]. In that work we considered asymptotically flat initial data for the vacuum Einstein equations which correspond to a Cauchy hypersurface of vanishing linear momentum. We showed that if the initial data satisfy a smallness condition then they give rise to a geodesically complete spacetime. We analyzed in detail the asymptotic behavior of the solutions at null and timelike infinity. The results which have to do with the behavior at null infinity, which is what concerns us here, are largely independent of the smallness condition which was introduced to ensure completeness. Among these results is the formula for the difference of the limits Σ^+ and Σ^- of the asymptotic shear Σ of outgoing null hypersurfaces C_u^+ as u tends to $+\infty$ and $-\infty$, respectively, which plays a crucial role in the present Letter. The rigorous derivation of this formula given in [4] relies heavily on the results developed in that work. For this reason we shall give below a simple derivation of the formula which is as much as possible self-contained.

Let S_0 be a spherical spacelike surface surrounding the source in a neighborhood of the intersection of the source with the boundary of the past of an event p of observation at the Earth, and lying in an asymptotically flat Cauchy hypersurface Σ_0 of vanishing linear momentum. Let C_0^+ be the outer boundary of the future of S_0. Denoting by B_0 the interior of S_0 in Σ_0, let, for each $d>0$, B_d be the set of points in Σ_0 whose distance from B_0 is less than d. We define $B^* = B_{d^*}$ to be the smallest region B_d containing the past of p in Σ_0. We then define C^{*-} to be the boundary of the domain of dependence of B^*. Then p lies in a neighborhood of the spherical spacelike surface S_0^* of intersection of C_0^+ with C^{*-}. We suppose that S_0 is chosen so that the generators of C_0^+ have no future end points.

Consider an arbitrary closed spacelike surface S in spacetime. We denote by γ the induced metric on S and by $d\mu_\gamma$, $\nabla\!\!\!/$, and K, the area element, covariant derivative, and Gauss curvature of γ, respectively. We define the radius r of S by $r=\sqrt{A/4\pi}$, where A is the area of S. Let l and $\underline{l}$ be, respectively, outgoing and incoming future-directed null normal vector fields to S subject to the normalization condition $g(l,\underline{l})=-2$. Then l and $\underline{l}$ are unique up to the transformation $l\rightarrow al$, $\underline{l}\rightarrow a^{-1}\underline{l}$, where a is a positive function on S. The null second fundamental form χ and the conjugate null second fundamental form $\underline{\chi}$ of S are two-covariant symmetric tensor fields on S defined by $\chi(X,Y)=g(\nabla_X l,Y)$, $\underline{\chi}(X,Y)=g(\nabla_X \underline{l},Y)$ for any pair of vectors X,Y tangent to S at a point. We denote by $\hat{\chi}$ and $\hat{\underline{\chi}}$ the trace-free parts of χ and $\underline{\chi}$, respectively. The torsion ζ of S is the one form on S defined by $\zeta(X)=\frac{1}{2}g(\nabla_X l,\underline{l})$ for any vector X tangent to S at a point. The mass aspect function μ and the conjugate mass aspect function $\underline{\mu}$ of S are functions on S defined by $\mu=K+\frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi}-\operatorname{d\!\!\!/iv}\zeta$, $\underline{\mu}=K+\frac{1}{4}\operatorname{tr}\chi\operatorname{tr}\underline{\chi}+\operatorname{d\!\!\!/iv}\zeta$. Also the spacetime curvature at S decomposes into the two-covariant symmetric tensor fields $\alpha,\underline{\alpha}$, the one-forms $\beta,\underline{\beta}$, and the functions ρ,σ on S, given by $\alpha(X,Y)=R(X,l,Y,l)$, $\underline{\alpha}(X,Y)=R(X,\underline{l},Y,\underline{l})$, $\beta(X)=\frac{1}{2}R(X,l,\underline{l},l)$, $\underline{\beta}(X)$

$=\frac{1}{2}R(X,\underline{L},\underline{L},l)$, $\rho=\frac{1}{4}R(\underline{L},l,\underline{L},l)$, $\sigma\epsilon(X,Y)=\frac{1}{2}R(X,Y,\underline{L},l)$, where X,Y are arbitrary vectors tangent to S at a point and ϵ is the area two-form of S. If $\mathrm{tr}\chi\mathrm{tr}\underline{\chi}<0$ we can fix l and $\underline{l}$ by requiring that $\mathrm{tr}\chi+\mathrm{tr}\underline{\chi}=0$. With this choice, the unit timelike normal to S given by $T=\frac{1}{2}(l+\underline{l})$ is the binormal of S. The Hawking mass m of S is given by

$$m=(r/2)\left[1+(1/16\pi)\int_S \mathrm{tr}\chi\mathrm{tr}\underline{\chi}\,d\mu_\gamma\right]$$

and is independent of the ambiguity in the choice of l and $\underline{l}$. If S has the topology of S^2, then by the Gauss-Bonnet formula we have $\int_S\mu\,d\mu_\gamma=\int_S\underline{\mu}\,d\mu_\gamma=8\pi m/r$.

We now return to the spherical surface $S_0^*=C_0^+\cap C^{*-}$. We choose l and $\underline{l}$ on S_0^* so that $\frac{1}{2}(l+\underline{l})=T$ is the binormal of S_0^*. We then extend $\underline{l}$ to C^{*-} to be the null geodesic vector field whose integral curves are the generators of C^{*-} and we consider on C^{*-} the level surfaces of the affine distance from S_0^*. We then define the retarded time function u on C^{*-} by the requirement that on each level surface, $u=2(r_0^*-r^*)$, where r^* is the radius of the level surface and r_0^* is the radius of S_0^*. We use the value of u to label a level surface S_u^*. The vector field l then extends to C^{*-} by the condition that on each S_u^*, l is the unique outgoing future-directed null normal to S_u^* conjugate to $\underline{l}$. Let ϕ_0 be a diffeomorphism of S^2 onto S_0^* and let, for each u, ϕ_u be the diffeomorphism of S^2 onto S_u^* such that $\phi_u\circ\phi_0^{-1}$ is the diffeomorphism of S_0^* onto S_u^* given by the flow of $\underline{l}$ on C^{*-}. If w is a p-covariant tensor field in spacetime we then have $2(\partial/\partial u)(\phi_u^*w)=\phi_u^*(\underline{D}w)$, where $\underline{D}w$ denotes the projection to S_u^* of the Lie derivative of w with respect to $\underline{l}$. Our entire discussion relies on the fact that the radius of S_0 is negligible in comparison to the radius of S_0^*. To make precise limit statements, we think of d^* as tending to ∞ so that the region B^* exhausts Σ_0. Then S_0^* moves to the infinite future along C_0^+ and its radius $r_0^*\to\infty$. As this happens the image by ϕ_0 of each point on S^* is to trace a generator of C_0^+. Then $\phi_0^*(r^{-2}\gamma)$, the pullback to S^2 of the induced metric on S_0^* rescaled by $r^{-2}=r_0^{*-2}$, converges to a metric γ^0 on S^2 Gauss curvature $K^0=1$. The metric γ^0 is therefore isometric to the standard metric on S^2. In fact, for each fixed u, $\tilde{\gamma}=\phi_u^*(r^{-2}\gamma)$, the pullback to S^2 of the induced metric on S_u^* rescaled by $r^{-2}=r^{*-2}$, converges to γ^0, and $\tilde{\nabla}$, the covariant derivative of $\tilde{\gamma}$, converges to ∇^0, the covariant derivative of γ^0, while $\phi_u^*(r^2/K)$, the Gauss curvature of $\phi_u^*(r^{-2}\gamma)$, converges to 1. Furthermore, for each fixed u, $\phi_u^*(r\,\mathrm{tr}\chi)\to 2$ and $\phi_u^*(r\,\mathrm{tr}\underline{\chi})\to -2$ as r_0^* and therefore r^* tends to infinity. If we define on each S_u^* the function $h=r\,\mathrm{tr}\chi-2$, then for each fixed u, $\phi_u^*(rh)$ tends to a function H on S^2. On the other hand, $\phi_u^*\hat{\chi}$ and $\phi_u^*(r^{-1}\hat{\underline{\chi}})$ tend to Σ and Ξ, respectively, two-covariant symmetric tensor fields on S^2 which are trace-free relative to γ^0. Also, $\phi_u^*(r\zeta)$ tends to Z, a one-form on S^2, and $\phi_u^*(r^3\mu)$ and $\phi_u^*(r^3\underline{\mu})$ tend to functions N and $\underline{N}$ on S^2, respectively, while $m(S_u^*)$ converges to $M(u)$, the Bondi mass. Moreover, by the main theorem of [4], for each fixed u, $\phi_u^*(r^{-1}\underline{\alpha})$, $\phi_u^*(r\underline{\beta})$, $\phi_u^*(r^3\rho)$, and $\phi_u^*(r^3\sigma)$ converge to A, B, P, and Q, respectively, as $r_0^*\to\infty$, while $|\alpha|,|\beta|=O(r^{*-7/2})$. Here A is a two-covariant symmetric tensor field on S^2 which is trace-free relative to γ^0, B is a one-form, and P and Q are functions on S^2.

The null Codazzi equation and its conjugate read

$$\mathrm{d\!/\!iv}\,\hat{\chi}+\hat{\chi}\cdot\zeta=\tfrac{1}{2}(\nabla\mathrm{tr}\chi+\mathrm{tr}\chi\zeta)-\beta\,,$$

$$\mathrm{d\!/\!iv}\,\hat{\underline{\chi}}-\hat{\underline{\chi}}\cdot\zeta=\tfrac{1}{2}(\nabla\mathrm{tr}\underline{\chi}-\mathrm{tr}\underline{\chi}\zeta)+\underline{\beta}\,.$$

Multiplying the first equation by r^2 and the second by r, pulling them back to S^2 by ϕ_u, and taking the limit $r_0^*\to\infty$, we obtain, making use of the above results, the following equations on S^2:

$$\mathrm{d\!/\!iv}^0\Sigma=\tfrac{1}{2}\nabla^0H+Z\,, \tag{1}$$

$$\mathrm{d\!/\!iv}^0\Xi=B\,. \tag{2}$$

In view of the Gauss equation of S_u^*, $K=\frac{1}{4}\mathrm{tr}\chi\mathrm{tr}\underline{\chi}-\frac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}}=-\rho$, and the definition of $\underline{\mu}$, the torsion of S_u^* satisfies the Hodge system:

$$\mathrm{c\!/\!url}\,\zeta=\sigma-\tfrac{1}{2}\hat{\chi}\wedge\hat{\underline{\chi}},\quad \mathrm{d\!/\!iv}\,\zeta=\underline{\mu}+\rho-\tfrac{1}{2}\hat{\chi}\cdot\hat{\underline{\chi}}\,.$$

The limit as $r_0^*\to\infty$ of this system rescaled by r^3 is the following Hodge system on S^2:

$$\mathrm{c\!/\!url}^0Z=Q-\tfrac{1}{2}\Sigma\wedge\Xi,\quad \mathrm{d\!/\!iv}^0Z=\underline{N}+P-\tfrac{1}{2}\Sigma\cdot\Xi\,. \tag{3}$$

Along C^{*-}, $\mathrm{tr}\chi$ propagates according to

$$\underline{D}\,\mathrm{tr}\chi+\tfrac{1}{2}\mathrm{tr}\underline{\chi}\mathrm{tr}\chi=-2\mu+2|\zeta|^2\,.$$

The limit of the corresponding propagation equation for rh is, simply,

$$\frac{\partial H}{\partial u}=0\,. \tag{4}$$

The propagation along C^{*-} of $\hat{\chi}$ obeys

$$\hat{\underline{D}}\hat{\chi}-\tfrac{1}{2}\mathrm{tr}\underline{\chi}\hat{\chi}=-\tfrac{1}{2}\mathrm{tr}\chi\hat{\underline{\chi}}+\hat{\mathcal{L}}_\zeta\gamma+2\zeta\zeta-|\zeta|^2\gamma\,,$$

where the caret denotes the operation of taking the trace-free part of a two-covariant symmetric tensor on S_u^* and $\mathcal{L}_\zeta$ denotes the Lie derivative with respect to the vector field corresponding to the one-form ζ. Taking the limit $r_0^*\to\infty$ we obtain

$$2\frac{\partial\Sigma}{\partial u}=-\Xi\,. \tag{5}$$

The propagation law of $\hat{\underline{\chi}}$ along C^{*-}, $\hat{\underline{D}}\hat{\underline{\chi}}=-\underline{\alpha}$, rescaled by r^{-1}, becomes, in the limit $r_0^*\to\infty$,

$$2\frac{\partial\Xi}{\partial u}=-A\,. \tag{6}$$

Finally, the propagation law of $\underline{\mu}$ along C^{*-},

$$\underline{D}\underline{\mu}+\tfrac{3}{2}\mathrm{tr}\underline{\chi}\underline{\mu}=-\tfrac{1}{4}\mathrm{tr}\chi|\hat{\underline{\chi}}|^2+\tfrac{1}{2}\mathrm{tr}\underline{\chi}|\zeta|^2 -2\mathrm{d\!/\!iv}(\hat{\underline{\chi}}\cdot\zeta)-\nabla\mathrm{tr}\underline{\chi}\cdot\zeta\,,$$

rescaled by r^3, yields in the limit the asymptotic propagation law

$$2\,\partial\underline{N}/\partial u = -\tfrac{1}{2}|\Xi|^2 . \tag{7}$$

Since we have $\bar{N}=\bar{\underline{N}}=2M$ (the overbar denotes mean value on S^2), the integral of (7) over S^2 is the Bondi mass loss formula [5]:

$$\frac{\partial M}{\partial u} = -\frac{1}{32\pi}\int_{S^2}|\Xi|^2 d\mu_{\gamma^0} . \tag{8}$$

According to the main theorem of [4] we have $A=O(|u|^{-5/2})$, $B=O(|u|^{-3/2})$ as $|u|\to\infty$. Equation (2) then implies that $\Xi=O(|u|^{-3/2})$ as $|u|\to\infty$. Therefore by (5), Σ tends to limits Σ^+ and Σ^- as $u\to+\infty$ and $u\to-\infty$, respectively, and $\Sigma^+-\Sigma^- = -\frac{1}{2}\int_{-\infty}^{+\infty}\Xi\,du$. Let us define on S^2 the function

$$F=\tfrac{1}{8}\int_{-\infty}^{+\infty}|\Xi|^2 du . \tag{9}$$

In view of (8), $F/4\pi$ is the total energy radiated to infinity in a given direction, per unit solid angle. By (7), $\underline{N}$ tends to limits $\underline{N}^+$ and $\underline{N}^-$ as $u\to+\infty$ and $u\to-\infty$, respectively, and we have $\underline{N}^+-\underline{N}^-=-2F$. The integrals of Eqs. (3) on S^2 are

$$\bar{Q}=\tfrac{1}{2}\overline{\Sigma\wedge\Xi},\quad \bar{P}=-\bar{\underline{N}}+\tfrac{1}{2}\overline{\Sigma\cdot\Xi} .$$

According to the main theorem of [4], under the hypotheses stated therein, we have $P-\bar{P},Q-\bar{Q}=O(|u|^{-1/2})$ as $|u|\to\infty$; therefore the limits $(P-\bar{P})^+$, $(P-\bar{P})^-$, $(Q-\bar{Q})^+$, and $(Q-\bar{Q})^-$ all vanish. Now while this is always true for the second pair, it is true for the first pair only in the case that the final center-of-mass frame is at rest relative to the initial center-of-mass frame and the initial and final velocities of the masses in the corresponding frames are negligible. In general $(P-\bar{P})^+$ and $(P-\bar{P})^-$ are determined from sums of boosted Schwarzschild solutions. For example, in the case of two bodies initially in nonrelativistic motion, which coalesce to one body of final rest mass M^+ and with final (recoil) velocity V relative to the initial center-of-mass frame, we have $(P-\bar{P})^-=0$ while at $\xi\in S^2\subset\Re^3$,

$$(P-\bar{P})^+(\xi) = -2M^+\left[\frac{(1-|V|^2)^{3/2}}{(1-\langle\xi,V\rangle)^3}-\frac{1}{(1-|V|^2)^{1/2}}\right].$$

Making use of the results above we conclude from the system (3) that Z tends to limits Z^+ and Z^- as $u\to+\infty$ and $u\to-\infty$, respectively, and Z^+-Z^- satisfies the Hodge system:

$$\not{\mathrm{curl}}^0(Z^+-Z^-)=0 ,$$
$$\not{\mathrm{div}}^0(Z^+-Z^-)=(P-\bar{P})^+-(P-\bar{P})^- +\underline{N}^+-\underline{N}^--\bar{\underline{N}}^+ +\bar{\underline{N}}^- .$$

Hence, defining the function Φ to be the solution, of vanishing mean, of the equation

$$\not{\Delta}^0\Phi=(P-\bar{P})^+-(P-\bar{P})^--2(F-\bar{F}) \tag{10}$$

on S^2, we have

$$Z^+-Z^-=\not{\nabla}^0\Phi , \tag{11}$$

while by (1), in view of (4), the difference $\Sigma^+-\Sigma^-$ satisfies the equation

$$\not{\mathrm{div}}^0(\Sigma^+-\Sigma^-)=Z^+-Z^- . \tag{12}$$

The above three equations determine $\Sigma^+-\Sigma^-$ uniquely. Their integrability condition is that Φ has vanishing projection $\Phi_{(1)}$ on the first eigenspace of $\not{\Delta}^0$. Thus,

$$P^+_{(1)}-P^-_{(1)}-2F_{(1)}=0 , \tag{13}$$

which expresses the law of conservation of linear momentum. In the example of binary coalescence, $P^+_{(1)}=-6\langle\xi,V\rangle M^+/(1-|V|^2)^{1/2}$, $P^-_{(1)}=0$, and (13) says that the recoil momentum is equal and opposite to the momentum carried off by the radiation. The solution $(\Sigma^+-\Sigma^-)(X,Y)$, of (10)-(12), evaluated at an arbitrary pair X,Y of vectors in $\Re^3$ tangent to S^2 at ξ, is the sum of a "linear" contribution from $(P-P_{[1]})^+-(P-P_{[1]})^-$ (the subscript [1] denotes projection on the sum of the zeroth and first eigenspaces of $\not{\Delta}^0$, the projection on the zeroth eigenspace being the mean value), which has long been known (in a different form) in the slow motion limit [3], and which in the example of the binary coalescence is given by

$$-\frac{2M^+}{(1-|V|^2)^{1/2}}\frac{\langle X,V\rangle\langle Y,V\rangle-\frac{1}{2}\langle X,Y\rangle|\Pi V|^2}{1-\langle\xi,V\rangle}$$

(Π is the projection to plane orthogonal to ξ), and a "nonlinear" contribution from $F-F_{[1]}$ which is equal to

$$\frac{-1}{2\pi}\int_{|\xi'|=1}(F-F_{[1]})(\xi')\frac{\langle X,\xi'\rangle\langle Y,\xi'\rangle-\frac{1}{2}\langle X,Y\rangle|\Pi\xi'|^2}{1-\langle\xi,\xi'\rangle}d\mu_{\gamma^0}(\xi') .$$

When matter (i.e., electromagnetic or neutrino) radiation is present then if T is the energy tensor of matter, $\phi_u^*(r^2\frac{1}{4}T(\underline{l},\underline{l}))$ tends to a limit E as $r_0^*\to\infty$ and in (7)-(9) $|\Xi|^2$ is replaced by $|\Xi|^2+32\pi E$.

We now turn to discuss a laser interferometer gravitational-wave detector. This consists of a reference mass m_0 and two test masses m_1 and m_2 initially at rest on a plane, with m_1 and m_2 at equal distances d_0 and at right angles from m_0. The masses are free or are suspended by pendulums according to whether the experiment is performed in space or on the Earth's surface. In any case, for time intervals much shorter than the period of each of the suspending pendulums the motion of the masses on the horizontal plane can be considered free. Any difference in the light travel times between m_0 and m_1

and m_2, respectively, results in a difference of phase of the laser light at m_0. We may think of m_0 as describing a timelike geodesic Γ_0 in spacetime while m_1 and m_2 describe neighboring timelike geodesics Γ_1 and Γ_2, respectively. The mass m_0 defines a local inertial frame as follows. With T being the unit future-directed tangent vector field of Γ_0 and t the proper time along Γ_0, let, for each t, H_t be the spacelike geodesic hyperplane through the point $\Gamma_0(t)$ orthogonal to T. Choosing an orthonormal basis (E_1,E_2,E_3) at a point on Γ_0 orthogonal to T, we parallel propagate this basis along Γ_0. Then each point p in a neighborhood of Γ_0 is assigned coordinates (t,x^1,x^2,x^3), where t corresponds to the geodesic hyperplane H_t on which p lies and (x^1,x^2,x^3), determine the vector $X=x^1E_1+x^2E_2+x^3E_3$ which is the tangent vector at $\Gamma_0(t)$ of the spacelike geodesic generator of H_t along which p is found at parameter distance 1. The distance of p from Γ_0 is the corresponding arc length $d=[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$. The deviation of the metric components in this coordinate system from those of the corresponding Minkowski metric are of $O(Rd^2)$, where R denotes curvature components at Γ_0. Therefore the deviation from unity of the ratios of the distances d_1 and d_2 of m_1 and m_2 from m_0 to the corresponding light travel times is of $O(Rd_0^2)$. Now the fractional change $\Delta d/d_0$ of the distances d_1 and d_2 due to the passage of a gravitational wave shall be of $O(R\tau^2)$, where τ is the time scale over which the curvature varies significantly. Thus if the ratio d_0/τ is assumed to be small, differences in light travel time accurately reflect differences in distance. Furthermore, the same assumption allows us to replace the geodesic equation for Γ_1 and Γ_2 by the Jacobi equation: $d^2x^i/dt^2=R_{iTjT}x^j$, where $R_{iTjT}(t)$ are the components of the curvature along Γ_0 in the frame (T,E_1,E_2,E_3). We choose the vectors E_1 and E_2 in the direction of the masses m_1 and m_2 initially. The (x^1,x^2) plane is then the horizontal plane and the initial conditions are $x^i_{(A)}(-\infty)=d_0\delta^i_A$, $\dot{x}^i_{(A)}(-\infty)=0$, where we denote by x^i_A the coordinates of the test mass m_A, $A=1,2$. Suppose for simplicity that the source is in the direction of E_3, namely, the vertical. Then the horizontal plane is tangent to the surfaces S^*_u and the null normals l and $\underline{l}$ coincide with the vectors $T-E_3$ and $T+E_3$, respectively. Our results on the asymptotic behavior of the curvature components then imply that $\ddot{x}^3_{(A)}=O(r^{-2})$, $\ddot{x}^A_{(C)}=-\frac{1}{4}r^{-1}A_{AB}x^B_{(C)}+O(r^{-2})$. Here, A_{AB} are the components of A in the frame $(\tilde{E}_1,\tilde{E}_2)$ on S^2 defined by $rE_A=\phi^t_u\tilde{E}_A$. This frame is orthonormal relative to the metric $\tilde{\gamma}$, and hence, in the limit $r_0^*\to\infty$, also relative to γ^0. In view of the initial conditions, we have, to leading order in r^{-1}, $x^3_{(A)}=0$; therefore the motion is confined to the horizontal plane. Also, since the fractional displacements shall be negligible, we can replace the coordinates on the right-hand side by their initial values to obtain $\ddot{x}^A_{(B)}=-(d_0/4r)A_{AB}$. Now, the retarded time u can be identified with the proper time t along Γ_0. Therefore, integrating once and using (6) and the fact that $\Xi\to 0$ as $u\to-\infty$ we obtain $\dot{x}^A_{(B)}=(d_0/2r)\Xi_{AB}$, where Ξ_{AB} are the components of Ξ in the frame $(\tilde{E}_1,\tilde{E}_2)$. The fact that $\Xi\to 0$ as $u\to+\infty$ implies that the test masses return to rest after the passage of the gravitational wave. On the other hand, integrating again and using (5), the displacements of the test masses at proper time t from their original positions are given by

$$x^A_{(B)}(t)-x^A_{(B)}(-\infty)=-(d_0/2r)[\Sigma_{AB}(t)-\Sigma^-]\,.$$

Taking the limit $t\to\infty$ we conclude that the test masses suffer permanent displacements given by

$$\Delta x^A_{(B)}=-(d_0/r)(\Sigma^+_{AB}-\Sigma^-_{AB})\,. \qquad (14)$$

The difference on the right-hand side of (14) contains a nonlinear effect, determined, as we have shown above, by the function F which is quadratic in Ξ. Now the maximal value of $|\Sigma(t)-\Sigma^-|$ is of the order of the maximal kinetic energy of the bodies constituting the source, while F, and therefore also $|\Sigma^+-\Sigma^-|$, is of the order of the total radiated energy. Consequently for a binary coalescence the nonlinear permanent displacement is of the same order of magnitude as the maximal displacement during the passage of the wave and builds up over the time scale over which the energy is radiated.

The author wishes to thank K. S. Thorne and J. P. Ostriker for help in the exposition. This work is an outgrowth of a collaboration with Sergiu Klainerman. The author is supported in part by NSF Grant No. PHY 9006681.

[1] V. B. Braginsky and L. P. Grishchuk, Zh. Eksp. Teor. Fiz. **89**, 744 (1985) [Sov. Phys. JETP **62**, 427 (1986)].
[2] V. B. Braginsky and K. S. Thorne, Nature (London) **327**, 123 (1987).
[3] Ya. B. Zel'dovich and A. G. Polnarev, Astron. Zh. **51**, 30 (1974) [Sov. Astron. **18**, 17 (1974)].
[4] D. Christodoulou and S. Klainerman, "The Global Nonlinear Stability of Minkowski Space," Ann. Math. Stud. (to be published).
[5] H. Bondi, and M. G. J. van Burg, A. W. K. Metzner, Proc. Roy. Soc. London A **269**, 21–48 (1962).