Asymptotic Estimate of the n-Loop QCD Contribution to the Total e^+e^- Annihilation Cross Section

Geoffrey B. West

Theoretical Division, T-8, Los Alamos National Laboratory, MS B285, Los Alamos, New Mexico 87545

(Received 19 April 1991)

A technique for estimating the large-order QCD perturbative contributions to the total e^+e^- total cross section is presented which is based upon the renormalization group and momentum analyticity. Our estimate for the coefficient of $(\alpha_s/\pi)^3$ has recently been confirmed by two exact calculations. We find that the effective expansion parameter is, in fact, not α_s/π but rather $4\pi^2 e b_1(\alpha_s/\pi)$, where b_1 is the first coefficient in the expansion of the β function.

PACS numbers: 13.65.+i, 11.10.Jj, 12.20.Ds, 12.38.Bx

Asymptotic freedom dictates that perturbation theory governs the high-energy behavior of the total hadronic e^+e^- annihilation cross section [1]. Indeed, at sufficiently high energies this leads to the well-known and important result that R, the ratio of this cross section to that for $\mu^+\mu^-$ pair production, is given by the naive pointlike parton-model result, namely $\sum Q_i^2$, Q_i being the charge of the ith quark species. Leading corrections to this can be systematically and reliably calculated using QCD perturbation theory:

$$R[q^2/\mu^2,\alpha_s(\mu)] \approx \left(\sum Q_i^2\right) \sum_{n=0}^{\infty} r_n(q^2/\mu^2)(\alpha_s/\pi)^n . \quad (1)$$

Here, α_s is the conventional QCD fine-structure constant normalized at the arbitrary scale μ and q the four-momentum delivered by the e^+e^- pair. The first two coefficients in this expansion are easy to obtain; furthermore they are both scheme and q^2 independent: $r_0 = r_1 = 1$. The first nontrivial correction is r_2 which was calculated some ten years ago by three independent groups [2]. In the modified-minimal-subtraction ($\overline{\text{MS}}$) scheme with five flavors they found $r_2(1) = 1.41$. Although this gives a modest correction to R, its effect on the extraction of α_s and Λ, the QCD scale parameter, are quite significant.

About two years ago the three-loop contribution was first calculated [3] and found to give $r_3(1) = 64.9$. This is a remarkable result since it implies that the three-loop contribution actually exceeds that of the two-loop one. Furthermore, it decreases the estimate of Λ, as extracted from the data, by roughly a factor of 2. Since R plays such a central role in our understanding of QCD, being in many ways the cleanest prediction of the theory, it is important to understand the origin of the coefficients and whether it is reasonable to expect large numbers. This is particularly true since it is known that the series diverges [4,5]. In principle, a very accurate measurement of R is the least ambiguous method for measuring α_s and uncovering new degrees of freedom. Thus, some reasonable estimate of the higher-order contributions is certainly desirable; their precise value may be of less significance than having some reliable, though rough, estimate, especially since R is only measured to a few percent at best. Thus all one really needs to know is whether, for example, $r_3 \sim 5$ or ~ 50. Obviously, the brute force evaluation of all of the Feynman diagrams contributing in each order rapidly becomes an impossible task even with sophisticated computer algorithms; (already the evaluation of r_3 involves hundreds of graphs). Furthermore, when accomplished, their significance may be questionable. Some alternative technique is required.

Last year this problem was addressed [5], the results of which led to the asymptotic estimate

$$r_3(1) \approx -12 , \quad (2)$$

which is almost a factor of 6 smaller than the explicit calculation of Ref. [3]. The analytic expression for the estimate of $r_n(1)$ is given explicitly in Eq. (22) below. Very recently the explicit calculations have been repeated [6] by two independent groups (one being the original authors of Ref. [3]), both obtaining $r_3(1) = -12.8$ in remarkable agreement with our estimate above. This more or less precise agreement, though gratifying, should not be taken too seriously since our estimate is probably only accurate up to corrections of $O(b_2/nb_1^2)$, the b_i being coefficients in the expansion of the β function [see Eq. (6) below]. Conservatively this means that the estimate (2) could have as much as a 20% error though it is probably smaller than this, as discussed below. What should be taken seriously though, is that the result, Eq. (2), even with a 20% error, was in serious disagreement with the original calculation.

It is the intent of this paper to present the essence, and some of the details, of how the formula, Eq. (22), and consequently, the result, Eq. (2), is derived. Our strategy is to exploit two general features of the full theory that remain valid in the perturbative sector, namely, renormalizability and analyticity. The renormalization group (RG) dictates that the polarization function $\Pi(q^2/\mu^2,\alpha_s)$, whose absorptive part is proportional to R, is, in fact, not a function of q^2 and α_s separately [as is the case for individual Feynman graphs and, therefore, for each term in the perturbative series, Eq. (1)] but, rather, is a function of a single variable $(q^2/\mu^2)\exp(1/4\pi b_1\alpha_s + \cdots)$. Conse-

quently, analyticity in q^2 does not peacefully coexist with analyticity in α_s; thus the perturbation expansion as expressed in Eq. (1) is at odds with renormalizability and q^2 analyticity. The resolution of this potential problem is that the series diverges so that Π is in fact not analytic in α_s at $\alpha_s=0$. By judiciously exploiting the constraint that Π is to have a power-series expansion in α_s ultimately leads to the estimate Eq. (2). Most investigations of high-order estimates of perturbation theory have focused on nongauge theories and used as their starting point a path-integral representation [7]. The problem of extending these techniques to gauge theories has run into difficulties associated with ghosts, gauge invariance, and renormalizability. The latter is, of course, not explicitly encoded in the path integral; it leads to a sufficiently singular behavior in α_s that it is believed that the series is not Borel summable. This does not mean, however, that one cannot estimate the coefficients; indeed, in what follows, renormalizability plays a central role.

To begin, then, let us recall some standard definitions. The conventional photon polarization tensor is given by

$$\Pi_{\mu\nu}(q)=i\int d^4x\, e^{iqx}\langle 0|T[j_\mu(x)j_\nu(0)]|0\rangle \tag{3}$$

$$\equiv(q^2g_{\mu\nu}-q_\mu q_\nu)\Pi(q^2,\alpha_s)\,, \tag{4}$$

where $j_\mu(x)$ is the electromagnetic current operator. Its absorptive part is $\mathrm{Abs}\Pi=R/12\pi$. Both Π and R have perturbative expansions, as, for example, expressed in Eq. (1). Since R is a physical quantity it cannot depend on μ and so satisfies a homogeneous renormalization-group equation ($dR/d\mu=0$). In massless QCD this implies that R is expressible as a function of a single variable $(q^2/\mu^2)e^{2K(g)}\equiv z$:

$$R[q^2/\mu^2,\alpha_s(\mu)]=f(z)\,. \tag{5}$$

Here $g(\mu)$ is the renormalized gauge coupling ($g^2=4\pi\alpha_s$) and $K(g)\equiv\int^g dg'/\beta(g')$. In perturbation theory $\beta(g)\approx -g^3(b_1-b_2g^2+\cdots)$ so

$$2K(g)\approx 1/b_1g^2-b_2/b_1^2\ln(b_1/g^2+b_2)+\cdots\,. \tag{6}$$

Further terms in this expansion are analytic in g^2. Usually, the renormalization-group constraint is expressed in terms of the running coupling constant $\bar g(t)$ defined by $2K(\bar g)=2K(g)+t$, where $t=\ln q^2/\mu^2$. This is equivalent to the relationship $z=e^{2K(\bar g)}$. Equations (1) and (5) can be reexpressed in the form

$$R[q^2/\mu^2,\alpha_s(\mu)]=R[1,\bar\alpha_s(t)]$$

$$\approx\left(\sum Q_i^2\right)\sum_{n=0}^{\infty}r_n(1)[\bar\alpha_s(t)/\pi]^n\,, \tag{7}$$

so it is sufficient to estimate $r_n(1)$. As already mentioned, the first two coefficients are scheme (and momentum) invariant, the rest being scheme and q^2 dependent.

Since j_μ is a composite operator, an additional subtraction beyond the usual multiplicative ones needs to be invoked in order to render $\Pi_{\mu\nu}$ finite [2,8]. This subtraction is closely related to the subtraction that is required in writing a dispersion relation for Π which expresses its analytic properties in q^2. When $q^2\to\infty$, $R\sim\sum Q_i^2$, so a once-subtracted dispersion relation is required. To avoid dealing with this unknown subtraction, it has become conventional to work with the quantity $D(q^2/\mu^2,g^2)\equiv\partial\Pi(q^2/\mu^2,g^2)/\partial t$. This, like R, also satisfies a homogeneous RG equation and is therefore also expressible as a function of z alone. It satisfies the following representation:

$$D\left(\frac{q^2}{\mu^2},g^2\right)=\frac{q^2}{12\pi^2}\int_0^\infty\frac{dq'^2}{(q'^2-q^2)^2}R\left(\frac{q'^2}{\mu^2},g^2\right) \tag{8}$$

$$=\frac{q^2}{\mu^2}e^{2K(g)}\int_0^\infty\frac{dz}{12\pi^2}\frac{f(z)}{[z-(q^2/\mu^2)e^{2K(g)}]^2}\,. \tag{9}$$

Equation (9) thus incorporates two essential features of the theory, namely, renormalizability and analyticity in q^2, that are not explicitly encoded in the path integral. It will be convenient to express the perturbative expansion of D, analogous to (1), in the form

$$D(q^2/\mu^2,g^2)\approx\sum_{n=0}^{\infty}(-1)^n d_n(q^2/\mu^2)g^{2n}\,. \tag{10}$$

The RG can be used to relate the absorptive part of D to R in the following way:

$$\mathrm{Abs}D=\frac{g\beta(g)}{12\pi}\frac{\partial R}{\partial g^2}\,, \tag{11}$$

from which we can deduce ($n>1$)

$$r_n=-(-4\pi^2)^{n+1}\frac{3}{\pi nb_1}\left[\mathrm{Im}d_{n+1}+\frac{b_2}{b_1}\mathrm{Im}d_n+\cdots\right]. \tag{12}$$

Clearly, $\mathrm{Im}d_0=\mathrm{Im}d_1=0$. Notice also that, by substituting (11) into (7), one can derive the inverse of (12), namely, an equation for the d_n in terms of the r_n [9].

Our plan now is to try to use the representation (9) to glean useful information about the large-n behavior of the d_n and consequently of the r_n. To do so we first need to invert (10) so as to express the d_n in terms of D itself. To do so it is convenient to analytically continue in n by introducing a function $d(s,q^2/\mu^2)$ which reduces to d_n when $s=n$; s is to be considered a complex variable. Then the required formula is [5,10]

$$d(s,q^2/\mu^2)=\int_c\frac{dg^2}{2\pi i}(-g^2)^{-1-s}D\left(\frac{q^2}{\mu^2},g^2\right). \tag{13}$$

The contour can be chosen to wrap around the discontinuities in D. Substituting the representation (9) into (13) and interchanging orders of integration gives

$$d(s,q^2/\mu^2)=\int_0^\infty\frac{dz}{12\pi^2}f(z)\frac{\partial}{\partial t}\Phi(s,t,z)\,, \tag{14}$$

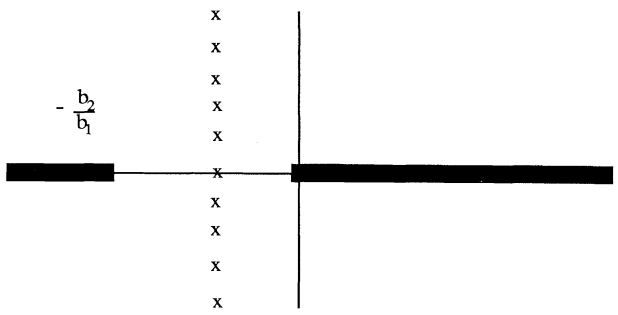

FIG. 1. Singularities of the function Φ, Eq. (16), in the $k \equiv 1/g^2$ plane. In the g^2 plane the string of poles lies along a loop accumulating at the origin with increasing density.

where

$$\Phi(s,t,z) \equiv \int_C \frac{dg^2}{2\pi i} \frac{(-g^2)^{-1-s}}{z - e^{t+2K(g)}} . \tag{15}$$

The singularity structure in the g^2 plane is complicated by an infinite sequence of poles which accumulate at the origin [11]. The situation is considerably simplified by transforming to the $k \equiv 1/g^2$ plane where

$$\Phi(s,t,z) = \int_{C'} \frac{dk}{2\pi i} \frac{(-k)^{s-1}}{z - e^{t+2K(k)}} . \tag{16}$$

The singularities in the k plane fall into three distinct categories as illustrated in Fig. 1: (i) a cut along the real positive axis needed to define $(-k)^{s-1}$ (this becomes impotent when s is an integer); (ii) an infinite sequence of almost equally spaced poles at $k \approx b_1[t + \ln z \pm 2\pi N i]$, $N = 0,1,2,\ldots$, arising from the vanishing of the denominator—the precise location of these poles is modified by higher-order terms in the β function [see Eq. (6)]; (iii) a cut at $-b_2/b_1$ needed to define $e^{K(k)}$ as a single-valued function—notice that higher-order coefficients in β do not affect the singularity structure in k. In the g^2 plane the sequence of poles lies along a closed curve whose density increases without bound as it approaches the origin. Furthermore, there is a potential essential singularity at $g^2 = 0$ arising from the e^K in the denominator.

The integral in (16) cannot be explicitly evaluated even in the case where only the first or second terms in β are retained [12]. However, when s becomes large we can use a saddle-point technique to estimate Φ. Saddle points occur when

$$k \equiv \frac{1}{g^2} = (s-1)\left[-\frac{\beta(g)}{g^3}\right][1 - ze^{-t-2K(g)}] . \tag{17}$$

For large s, the solutions to this fall into three classes reflecting the three classes of singularity: (i) $k \approx b_1(s-1) + b_2/b_1$; (ii) $2K(k) \approx t + \ln z \pm 2\pi N i$, corresponding to the string of poles, and (iii) $\beta(g)/g^3 \approx 0$ or $k \approx -b_2/b_1$, corresponding to the lip of the left-hand cut. Corrections to these are all of $O(1/s)$. In principle, the second two of these contributions could, of course, be evaluated directly using standard techniques. Suppose for the moment, however, we consider all contributions as approximated by Gaussian integrals; then the generic structure of our result will be of the form

$$d(s,q^2/\mu^2) \approx \sum_m \int_0^\infty \frac{dz}{12\pi^2} f(z) \frac{\partial}{\partial t} \frac{(-k_m)^{s-1}}{[2\pi\phi(k_m)]^{1/2}} \frac{1}{z - e^{t+2K(k_m)}} , \tag{18}$$

where $k_m(s,t,z)$ is the position of the mth saddle point and

$$\phi(k_m) \equiv \left(\frac{s-1}{k_m}\right)\left[\frac{K''(k_m)}{K'(k_m)} + 2K'(k_m) - \frac{(s-2)}{k_m}\right] . \tag{19}$$

(As usual, due care must be exercised in ascribing the correct phase structure to be associated with each k_m.) From this expression it is clear that, for large s, the dominant behavior is given by the contribution of the first saddle point at $k_1 \sim b_1(s-1)$ since this leads to an s^s type of behavior; the other saddle points (in actuality poles and cuts) lead only to an ordinary power behavior $\sim c^2$, where c is s independent. Another distinguishing and crucial property of the first saddle point is that its position does not depend on z. This means that its contribution to (18) can be reexpressed as

$$d(s,q^2/\mu^2) \approx \left[\frac{2}{\pi\phi(k_1)}\right]^{1/2} k_1^{s-1} D\left(\frac{q^2}{\mu^2}, \frac{1}{k_1}\right)\cos\pi s . \tag{20}$$

Using Eq. (11), one can now extract the leading behavior as $k_1 \sim b_1(s-1) \to \infty$:

$$\mathrm{Im} d(s,1) \approx -\frac{b_1}{24\pi^3} \frac{k_1^{s-3}}{[2\pi\phi(k_1)]^{1/2}} \cos\pi s . \tag{21}$$

Substituting this into Eq. (12) and reexpressing the result in terms of a gamma function gives

$$r_n(1) \approx -\frac{e^{1-b'}}{\pi} (4\pi^2 e b_1)^{n-1} \frac{\Gamma(n+b')}{(n+b')^2} , \tag{22}$$

where $b' \equiv b_2/b_1^2$. With five flavors this gives

$$r_3(1) \approx -13.4 , \tag{23}$$

in excellent agreement with the new exact calculations [6].

Various comments are in order: (a) This estimate is slightly larger than my original estimate [5] because I have tried to keep all terms of $O(b'/n)$. It is tedious to do this consistently since the corrections to the saddle-point

estimate and the neglected contributions are difficult to evaluate. The other saddle points give contributions that are down by terms of $O(c/n)^n$; furthermore, there are other exponential corrections $\sim e^{-n}$ which have been neglected and are expected to be small. The leading correction to (22) coming from the dominant saddle point (beyond the b'/n terms) is of $O(b_3/b_1^3 n^2)$. This correction represents the first appearance of scheme dependence in our estimate; it vanishes as $n \to \infty$ provided a scheme is chosen in which b_3, for example, is not "unnaturally" large. In $\overline{\text{MS}}$ with five flavors, $b_3/b_1^3 \approx 0.4$ so the scheme-dependent contribution is very small; one should compare $b_3/b_1^2 n^2 \approx 0.05$ to $b_2/b_1^2 n \approx 0.23$. Thus, implicit in the expression (22) is a choice of a wide class of schemes, satisfying the above constraint. In principle, one could formulate the problem in a scheme-invariant fashion and only evaluate, or estimate, scheme-invariant quantities.

(b) Equation (22) explicitly exhibits both the divergence of the series and the lack of Borel summability; indeed the r_n grow like $n!$ with

$$r_{n+1}/r_n \sim 4\pi^2 e b_1 n . \tag{24}$$

Our estimates for the next two coefficients are

$$r_4(1) \approx -158.6, \quad r_5(1) \approx -2609.8 . \tag{25}$$

A major question that will be addressed in later papers is how to make sense out of this uncontrollable growth in the coefficients [5], especially since the series is not Borel summable. Notice, incidentally, that Eq. (24) implies that the effective expansion parameter is not α_s/π but rather $4\pi^2 e b_1(\alpha_s/\pi) \approx 5.21(\alpha_s/\pi) \approx \frac{1}{4}$ (taking $\alpha_s \simeq 0.15$). Naively changing the sign of b_1 induces a factor $(-1)^n$ into r_n which would allow the possibility of Borel summing. This suggests that infrared-free theories ought to be summable.

(c) Notice that our estimate has an absolute normalization; in fact, it is given by the value of $d_1(1)$ which is unity.

[1] For a general overview and background see, e.g., T-P. Cheng and L-F. Li, *Gauge Theory of Elementary Particle Physics* (Oxford Univ. Press, New York, 1984).

[2] K. G. Chetyrkin, A. L. Kataev, and F. V. Tkachov, Phys. Lett. **85**, 277 (1979); W. Celmaster and R. Gonsalves, Phys. Rev. Lett. **44**, 560 (1980); M. Dine and J. Sapirstein, *ibid.* **43**, 668 (1979).

[3] S. G. Gorishny, A. L. Kataev, and S. A. Larin, Phys. Lett. B **212**, 238 (1988).

[4] This can be deduced from a generalization of Dyson's original argument which showed that the Feynman series in QED is divergent; see F. Dyson, Phys. Rev. **85**, 631 (1952).

[5] G. B. West, in *Radiative Corrections,* Proceedings of the NATO Workshop, July 1989, edited by N. Dombey and F. Boudjema (Plenum, New York, 1990), p. 487. The main aim of this paper was to try to understand the remarkable size of the coefficient quoted in Ref. [3] with little attention given to its sign. The sign was clarified in later presentations of this work.

[6] L. R. Surguladze and M. A. Samuel, Phys. Rev. Lett. **66**, 560 (1991); S. G. Gorishny, A. L. Kataev, and S. A. Larin, University of Michigan Report No. UM-TH-91-01 (to be published).

[7] See, e.g., J. Zinn-Justin, Phys. Rep. **70**, 109 (1981).

[8] G. B. West, Nucl. Phys. **B288**, 444 (1986).

[9] J. D. Bjorken, SLAC Report No. SLAC-PUB-5103 (to be published).

[10] This can also be derived by writing an appropriate dispersion relation for D in terms of the variable g^2.

[11] These are related to a corresponding infinite sequence of cuts for D itself which manifest themselves as renormalons in the Borel plane; see, G. 't Hooft, in *Deeper Pathways in High Energy Physics,* Proceedings of Orbis Scientiae, Coral Gables, 1977, edited by A. Perlmutter and L. F. Scott (Plenum, New York, 1977), p. 699.

[12] It is worth noting that, although Φ is a very complicated function, its structure is related to that of the generalized Riemann ζ function when only the first term in the β function is kept. Presumably this is related to the ubiquitous occurrence of ζ functions in the exact calculations of Ref. [6]. Indeed it might be possible to predict the ζ function structure in the exact expansion for r_n.