Supersymmetric Bianchi Type IX Cosmology

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The homogeneous cosmology of Bianchi type IX (mixmaster cosmology) is quantized in a supersymmetric form, allowing the Dirac-type square root of its Wheeler-DeWitt equation to be taken. A supersymmetric solution is given exactly in closed form.

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Homogeneous but anisotropic cosmological models of Bianchi type IX have long occupied an important place in theoretical cosmology. Such models were first introduced and studied in detail by Belinsky, Khalatnikov, and Lifshitz [1] in an attempt to exhibit in a spatially homogeneous model the generic form of the cosmological singularity implied by general relativity. Independently, Misner studied the Bianchi type IX model [2] and introduced the name "mixmaster cosmology" to describe its peculiar way of approaching the initial singularity in an infinitely oscillating and chaotic way.

In its classical form the model has served as an important example of "chaos" in general relativity [3]. In its quantized form, first studied by Misner and co-workers [4], the model served to illustrate the concept of "superspace" in a spatially homogeneous context called "minisuperspace." (We are referring here to the superspace of geometrodynamics.) Misner and co-workers have shown that asymptotically close to the initial singularity the model becomes equivalent to a certain triangular quantum billiard on a space of constant negative curvature whose level statistics was recently analyzed numerically in a separable approximation [5] and also in the general nonseparable case [6].

The quantized model has also been used as an example [7] to exhibit the consequences of the proposal by Hartle and Hawking [8] or other proposals [9] for the boundary condition distinguishing the physically realized solution of the Wheeler-DeWitt equation. In all studies of the quantized Bianchi type IX cosmologies the solutions given were either approximate or numerical. In the present paper we present a new approach to the quantization of the Bianchi type IX cosmology. We shall show that due to a hidden symmetry, which so far seems to have gone unnoticed, the quantization can be performed in a way which fully respects the given classical limit of the model while introducing a supersymmetry in its quantized version. It may be hoped that this supersymmetric model could also be derived directly from the full Lagrangian of supergravity. Such derivations have been given in the literature for the (spatially flat) Bianchi type I [10] and the isotropic Friedmann [11] cosmologies, which are contained formally as special cases in the Bianchi type IX model by putting the curvature and the anisotropy to zero, respectively. There remain of course significant differences between the Bianchi types I and IX (infinite versus finite volume) and between the Bianchi type IX and the closed Friedmann universe (existence versus nonexistence of classical vacuum solutions). However, these remaining differences are not important in the present context: Bianchi type I may be given a finite volume by imposing the topology of a 3-torus, and the closed Friedmann universe without matter appears quantum mechanically by vacuum fluctuations (see, e.g., [4]) just as the empty Bianchi type IX universe does (see below). The supersymmetric quantization provides the homogeneous universe with additional fermionic degrees of freedom. Nevertheless it leads to a striking simplification of the quantized system, whose Wheeler-DeWitt equation, restricted to the sector where the fermionic variables vanish, is exactly solvable for its supersymmetric "ground state," despite the fact that the classical limit is chaotic.

The homogeneous Bianchi type IX metric has the form

$$ds^2 = -N^2 dt^2 + (6\pi)^{-1/2} e^{2\alpha} (e^{2\beta})_{ij} \sigma^i \sigma^j , \qquad (1)$$

where N, α, and β_{ij} are functions of t only, and the σ^i are the 1-forms dual to the basis vectors on the three-sphere SO(3). N is the lapse function, which we shall choose as $N=1$, $(6\pi)^{-1/2}\exp(\alpha)$ is the scale factor of the universe, and the traceless diagonal matrix $\beta_{ij} = \mathrm{diag}(\beta_+ + \sqrt{3}\beta_-, \beta_+ - \sqrt{3}\beta_-, -2\beta_+)$ parametrizes the anisotropy of the spacelike cross sections $t=$const. For a discussion of the classical aspects of this cosmology we refer to [12], to whose notation we adhere as much as possible. Its "superspace Hamiltonian" H_0, satisfying the Hamiltonian constraint $H_0 = 0$, is given by

$$2H_0 = G^{\mu\nu} p_\mu p_\nu + U(q) , \qquad (2)$$

where the generalized coordinates $q^\nu = (\alpha, \beta_+, \beta_-)$ with $\nu = 0,1,2$ span the minisuperspace with the metric $G^{\mu\nu} = \mathrm{diag}(-1,1,1) = G_{\mu\nu}$, which we may choose as flat, making use of the fact that superspace is defined only up to conformal transformations [4], thereby fixing our choice of factor ordering, and where the p_ν are the canonically conjugate generalized momenta. The potential U can be written as $U = e^{4\alpha}[V(\beta_+, \beta_-) - 1]$, where

$$3V = \exp(-8\beta_+) + 2(\cosh 4\sqrt{3}\beta_- - 1)\exp(4\beta_+) - 4(\cosh 2\sqrt{3}\beta_-)\exp(-2\beta_+) + 3 .$$

It defines an apparently complicated potential well, depending explicitly on the timelike coordinate α, and presenting exponentially steep and moving potential walls in the (β_+,β_-) plane with triangular symmetry and narrow channels at the three corners, reaching infinity.

The key point we make in the present work is the observation that the potential $U(q)$, its complicated form notwithstanding, has a hidden symmetry beyond the triangular symmetry. This additional symmetry can be recognized by rewriting $U(q)$ in the form

$$U(q)=G^{\mu\nu}\frac{\partial\phi}{\partial q^\mu}\frac{\partial\phi}{\partial q^\nu}, \tag{3}$$

with $\phi=\frac{1}{6}e^{2\alpha}\mathrm{Tr}(e^{2\beta})$. It is easy to check that this is indeed a correct representation of U. The form (3) of the potential ensures that the Hamiltonian (2) is the "bosonic" part of a supersymmetric Hamiltonian [13]. Having uncovered this symmetry, it is natural to preserve it when quantizing the system. Thus the quantized superspace Hamiltonian is written in the form

$$H=\tfrac{1}{2}(Q\bar{Q}+\bar{Q}Q)=H_0+\frac{\hbar}{2}\frac{\partial^2\phi}{\partial q^\mu\partial q^\nu}[\bar\psi^\mu,\psi^\nu], \tag{4}$$

with the non-Hermitian supercharges

$$Q=\psi^\nu\left(p_\nu+i\frac{\partial\phi}{\partial q^\nu}\right),\quad \bar{Q}=\bar\psi^\nu\left(p_\nu-i\frac{\partial\phi}{\partial q^\nu}\right),$$

where the ψ^ν, $\bar\psi^\nu$ satisfy the spinor algebra

$$\begin{aligned}&\psi^\nu\psi^\mu+\psi^\mu\psi^\nu=0=\bar\psi^\nu\bar\psi^\mu+\bar\psi^\mu\bar\psi^\nu,\\ &\bar\psi^\nu\psi^\mu+\psi^\mu\bar\psi^\nu=G^{\mu\nu}.\end{aligned} \tag{5}$$

It follows that $Q^2=0=\bar{Q}^2$. The algebra (5) has an eight-dimensional matrix representation, which is equivalent to a representation in terms of three Grassmann variables η^ν and their derivatives $\bar\psi^\nu=\eta^\nu$, $\psi^\nu=G^{\nu\mu}\partial/\partial\eta^\mu$, which we shall now adopt. We note that the quantized H differs from the classical H_0 by a spin term, but this term vanishes in the classical limit. As H commutes with the "fermion number" $\eta^\nu\partial/\partial\eta^\nu$ we may decompose any solution of $H\Psi=0$ as

$$\Psi=A_++B_\nu\eta^\nu+\tfrac{1}{2}\varepsilon_{\nu\mu\lambda}C^\lambda\eta^\nu\eta^\mu+A_-\eta^0\eta^1\eta^2, \tag{6}$$

where the eight functions A_+, B_ν, C^ν, and A_- depend on the $q^\nu=(\alpha,\beta_+,\beta_-)$ only. Supersymmetric solutions must satisfy $Q\psi=0=\bar{Q}\psi$. They are obtained as

$$\begin{aligned}&A_\pm=a_\pm e^{\mp\phi(q)\hbar},\\ &B_\nu=\frac{\partial f_+(q)}{\partial q^\nu}e^{-\phi(q)/\hbar},\quad C^\nu=G^{\nu\mu}\frac{\partial f_-(q)}{\partial q^\mu}e^{+\phi(q)/\hbar},\end{aligned} \tag{7}$$

where $a_\pm$ are constants and $f_\pm(q)$ are functions satisfying the equations

$$G^{\nu\mu}\left(\hbar\frac{\partial}{\partial q^\nu}\mp2\frac{\partial\phi}{\partial q^\nu}\right)\frac{\partial f_\pm(q)}{\partial q^\mu}=0. \tag{8}$$

We note that for Bianchi type I we have $\phi=0$ and Eq. (8) admits plane-wave solutions, which have been found in [10]. For the present case, $\phi\neq0$ nontrivial solutions of Eq. (8), if they exist, are still waves in superspace of the asymptotic form $f_\pm\sim\exp(iS/\hbar)$, whose characteristics $(p_\nu=\partial S/\partial q^\nu)$ are constrained by $G^{\nu\mu}p_\nu p_\mu=0=G^{\nu\mu}p_\nu\partial\phi/\partial q^\mu$.

In the following we shall in any case concentrate on the solutions $A_\pm$ in the empty and the filled fermion sectors, respectively. In the special case of the Friedmann universe, where $\beta_+=\beta_-=0$, these are the only components of Ψ, and our result reduces to one given earlier for this case [11]. Without yet committing ourselves to a particular statistical interpretation of the wave function of the universe it seems reasonable to demand that Ψ does not diverge for $|\beta_\pm|\to\infty$ at fixed α. This rules out solutions $A_-\neq0$; i.e., we must fix $a_-=0$ (and also $f_-=0$).

It is interesting to note that the solution we find, if specialized to the case of the isotropic Friedmann universe $\beta_+=0=\beta_-$, is not the Hartle-Hawking state [8], which would be obtained from the function A_- (with $A_+=0$) we discarded due to the boundary conditions following from the presence of β_+,β_-. It should be noted, however, that the Hartle-Hawking state was calculated in [8] for a nonvanishing cosmological constant λ (and here the comparison is made for $\lambda\to0$ in the solution), which might change the state selected by their boundary condition. Let us also mention that an exact solution of the quantized Bianchi type IX model has also been claimed by Kodama [14] using nonconventional variables and not invoking supersymmetry. Because of the latter fact our

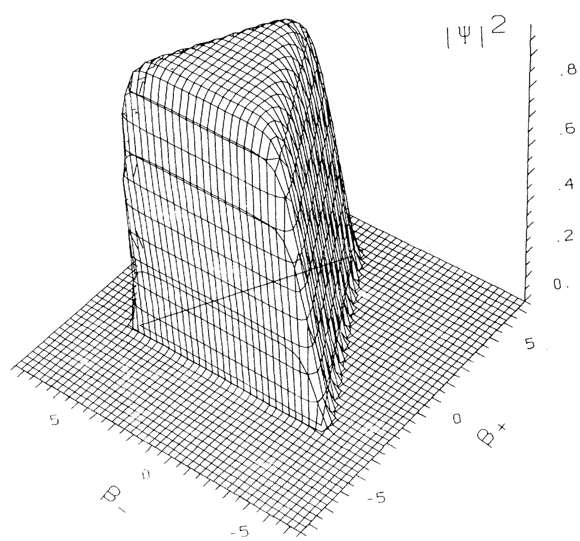

FIG. 1. $|\Psi|^2$ as a function of β_+ (right scale) and β_- (left scale) with its maximum normalized to 1 for fixed $\alpha=-5$, $\hbar=1$. The positions of the exponentially steep walls of ϕ on the equilateral triangle $2\beta_+-\alpha+\frac{1}{2}\ln3\hbar=0$ and sides rotated by $\pm120°$ around $\beta_+=\beta_-=0$ are also shown.

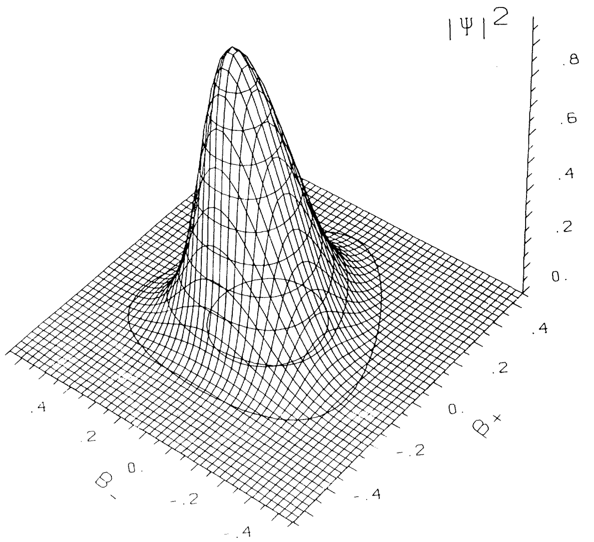

FIG. 2. $|\Psi|^2$ as in Fig. 1 for fixed $\alpha=1$, $\hbar=1$. Also shown as a circle is the mean-square fluctuation in Gaussian approximation.

solution cannot be equivalent to Kodama's, but so far a detailed comparison of the results in [14] and the work in [4] and [7] or the present work has resisted all efforts.

The statistical interpretation of the wave function, according to various proposals in the literature, can be made (i) by using $|\Psi|^2$ as an unnormalized probability density [8], (ii) by using the conserved Klein-Gordon-type density associated with the Wheeler-DeWitt equation [4,15], or (iii) by constructing a Dirac-type density conserved by the equation $Q\Psi=0$ [10]. According to (i), $P(\alpha,\beta_+,\beta_-)=\exp(-2\phi/\hbar)$ is the unnormalized probability density to observe given values of α,β_+,β_-. It decays very rapidly to zero for scale factors e^α larger than the Planck length, because there is neither a cosmological constant nor a matter field in the present model. $W(\alpha,\beta_+,\beta_-)=P/\int d\beta_+ d\beta_- P$ is the conditional probability density to observe given values of β_+,β_- at fixed α. It becomes infinitely broad for $\alpha\to-\infty$ as the cosmological singularity is approached, and extremely narrow for $\alpha>1$ where $\langle\beta_+^2\rangle\approx\langle\beta_-^2\rangle\approx e^{-2\alpha}/8$. In Figs. 1 and 2 we present a plot of this conditional probability density (in arbitrary units) over the (β_+,β_-) plane for $\alpha=-5$ and 1, respectively, in natural units $(\hbar=1)$. However, one may well question whether a statistical interpretation should be applied to the wave function in the present case, as the system never leaves the realm of quantum fluctuations and therefore remains unobservable for any conceivable definition of a measurement. Indeed, the probability density associated with the interpretation (ii) remains zero in the present case. It would be nonzero as soon as a tunneling out of the Planck regime became possible due to either the presence of a cosmological constant or a coupling to a massive field rendering the solution Ψ complex [9]. It would be very interesting to extend the present analysis by including any of these features, without destroying the supersymmetry. Finally, the interpretation (iii), somewhat unexpectedly, is not applicable in the present case, contrary to the Bianchi type I case [10], because $QA_+=0$, if solved for $\partial A_+/\partial\alpha$, yields $\hbar\,\partial A_+/\partial\alpha=-(\partial\phi/\partial\alpha)A_+$; i.e., it describes a nonunitary evolution of A_+ in the timelike coordinate α and a conserved probability density cannot be constructed.

In summary, we have exhibited a hidden supersymmetry of the mixmaster cosmology and preserved this symmetry in its quantized version. The supersymmetric state for vanishing Grassmann variables has been given exactly. It describes the zero-point fluctuations replacing the classical mixmaster oscillations.

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