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Rational Mappings, Arborescent Iterations, and the Symmetries of Integrability

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We describe a class of nonlinear birational representations of groups generated by a finite number of involutions. These groups are symmetries of the Yang-Baxter equations and their higher-dimensional generalizations. They provide discrete dynamical systems with a variety of behaviors, from chaotic to integrable, according to the number of invariants of the representation.

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The results we present here have their roots in statistical mechanics on d-dimensional lattices. They nevertheless go beyond this field and reach various domains: discrete dynamical systems, nonlinear representations of Coxeter groups, symmetries of the Yang-Baxter equations, and symmetry groups of phase diagrams, among others.

These results stand between the two extremes of the behavior of dynamical systems: integrability, which signals the existence of remarkable structures, but is exceptional, and chaotic behavior which accounts for many situations, but may reduce to an unsatisfactory taxonomy of a too large set of models.

We describe classes of infinite discrete groups, containing and generalizing the notion of iteration. These groups happen to be symmetries of the equations which nowadays define integrability, that is to say the Yang-Baxter equations and their higher-dimensional generalizations; see [1-4].

The generators of these groups are noncommuting involutions $I_1, I_2, \dots, I_r$. In our construction, the number r of involutions is originally $2, 4, \dots, 2^{d-1}$. *These involutions are simply different inversions of matrices, i.e., birational transformations with integer coefficients.*

If $d=2$, i.e., $r=2$, we have two involutions $I_1=I$ and $I_2=J$. The group Γ they generate is the set of transformations of the form $I^\alpha(IJ)^n$, where $\alpha=0,1$ and $n\in\mathbb{Z}$. Up to a semidirect product by $\mathbb{Z}_2$, this group identifies with the *iterations* of the map $f=IJ$. We get a discrete-time dynamical system, just as in the study *à la* Poincaré of dynamical systems [5].

If $d>2$, the number of generating involutions increases, and the group Γ has exponential growth. One has an *arborescent iteration.*

We produced images of the orbits of Γ. Remarkably enough these orbits sometimes are chaotic and sometimes lie in algebraic subvarieties of the parameter space. We may give the equations of these subvarieties by writing down the algebraic invariants of the transformations.

It is worth noticing that the mappings we construct are *not restricted to the interval.* We produce N-dimensional mappings for arbitrary N. These mappings admit natural multiparameter deformations [2], which nicely exemplify the breaking of tori *à la* Kolmogorov, Arnold, and Moser [6].

Construction of involutions.— We describe different construction procedures of inversions of matrices. They illustrate the variety of possible constructions but are of course not limitative.

(A) A first construction: Let m be a $q\times q$ matrix of entries m_{ij}. Let I be the ordinary matrix inverse. Let J be the element by element inverse ($m_{ij}\rightarrow 1/m_{ij}$). These two involutions do not commute, and IJ is of infinite order.

It is noticeable that certain patterns of matrices are preserved by the action of I and J separately, and therefore by the whole group Γ generated by I and J (e.g., symmetric or cyclic matrices). We may thus reduce the dimension of the space where I and J act. There exist patterns, defined by equalities between the entries (m_{ij}

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$=m_{kl}$ for certain pairs of indices), which are preserved simultaneously by I and, of course J. We call these patterns admissible. One has to realize their scarcity: There are, for instance, 187 such patterns for 4×4 matrices, to be compared with the total number of patterns $(10480142147 \simeq 10^{10})$. We have performed the exhaustive exploration of the possible patterns up to matrix size 4×4, and partial explorations for higher sizes. We have found admissible patterns for higher matrix sizes, for example, the noncyclic, nonsymmetric 6×6 matrix

$$m=\begin{pmatrix} x & y & z & y & z & z \\ z & x & y & z & y & z \\ y & z & x & z & z & y \\ y & z & z & x & z & y \\ z & y & z & y & x & z \\ z & z & y & z & y & x \end{pmatrix}. \tag{1}$$

These matrices form an Abelian algebra, on which two products are defined: the matrix product (denoted by $\cdot$) and the product element by element (denoted by $*$). There exists a *linear isomorphism* C between the two products:

$$C(M\cdot N)=C(M)*C(N). \tag{2}$$

This isomorphism defines a collineation between the two inverses:

$$I=C^{-1}JC. \tag{3}$$

(In $\mathbb{C}P_2$, the Noether theorem [7] proves that every birational automorphism of the plane can be represented as a product of quadratic transformations and a projective transformation, but this is very specific to $\mathbb{C}P_2$; the birational transformations in $\mathbb{C}P_n$, $n>2$, are much more complicated.)

Beware that C is not unique, and *is not of finite order.* However, in the so-called standard scalar Potts [8] limit $(y=z)$, *C is nothing but the Kramers-Wannier duality involution* [9].

We consider the entries of m as homogeneous coordinates in a projective space. The inverses I and J have a rational action on the inhomogeneous coordinates $u=y/x$, $v=z/x$. I reads

$$u\rightarrow \frac{-u^2-u+2v^2}{1+u+2v-u^2-2uv-v^2}, \tag{4}$$

$$v\rightarrow \frac{u^2+vu-v^2-v}{1+u+2v-u^2-2uv-v^2}, \tag{5}$$

J being simply $u\rightarrow 1/u$ and $v\rightarrow 1/v$.

The numerical iteration of IJ shows [1] that the successive images *all lie on curves.* These algebraic curves have the equation

$$\frac{(2v^2+2vu-u^2-2u^3-2vu^2+v^2u)(u-v^2)^2}{(v+u)^4(1-u)(1-v)^2}=\text{const}.$$

This rational expression is an invariant of I and J, and consequently of the whole group Γ. We have a linear pencil of curves of generic genus 1, with finitely many curves of genus 0.

(B) A second construction: Let m be a matrix of size $p^2\times p^2$. The entries of m may be written with double indices:

$$m_{kl}^{ij},\quad i,j,k,l\in\{1,\ldots,p\}.$$

On such matrices, there is a product law

$$(M\cdot N)_{kl}^{ij}=\sum_{\alpha\beta} M_{\alpha\beta}^{ij}N_{kl}^{\alpha\beta} \tag{6}$$

and an *inverse* I for this product, i.e.,

$$\sum_{\alpha\beta} M_{\alpha\beta}^{ij}\cdot(IM)_{kl}^{\alpha\beta}=\delta_k^i\delta_l^j. \tag{7}$$

The form of the entries allows for the definition of *partial transpositions* t_1 and t_2:

$$(t_1M)_{kl}^{ij}=M_{il}^{kj},\quad (t_2M)_{kl}^{ij}=M_{kj}^{il}. \tag{8}$$

The composition of the two partial transpositions is the full transposition t and commutes with the inverse I, *but the partial transpositions do not.* We may define

$$J=t_1It_2. \tag{9}$$

We have $J=t_2It_1$ and $J^2=1$. The two inversions I and J *do not commute.* They actually generate an infinite discrete group.

Remark.— We shall in the sequel use a generalization of this construction to multi-index matrices of size $p^d\times p^d$ written in the form $M_{j_1j_2\cdots j_d}^{i_1i_2\cdots i_d}$. There exist d different partial transpositions $t_1,t_2,\ldots,t_d$ with the evident definition

$$(t_kM)_{j_1\cdots j_k\cdots j_d}^{i_1\cdots i_k\cdots i_d}=M_{j_1\cdots i_k\cdots j_d}^{i_1\cdots j_k\cdots i_d}. \tag{10}$$

We clearly have a product and an inverse I for these multi-index matrices. We may define $2^{(d-1)}$ new inversions by

$$I_k=t_{\alpha_1}\cdots t_{\alpha_s}It_{\alpha_{s+1}}\cdots t_{\alpha_d}=t_{\alpha_{s+1}}\cdots t_{\alpha_d}It_{\alpha_1}\cdots t_{\alpha_s},$$

where $(\{\alpha_1,\ldots,\alpha_s\},\{\alpha_{s+1},\ldots,\alpha_d\})$ is a partition of $\{1,\ldots,d\}$. These various inverses are related by permutations of rows and columns. Note that the product of all t_k's is the full transposition t and commutes with all the inverses.

As for the first construction, we are lead to consider admissible patterns of matrices which are left invariant by the many inversions. This allows for a reduction of the number of homogeneous parameters the matrix depends on. An interesting example is given, in the case $d=2$, by the R matrix of the Baxter model [10]. The generic orbits again form in this case a dense set in elliptic curves (in $\mathbb{C}P_3$). *This resolves the problem of "Baxterization" of an isolated solution of the Yang-Baxter equations* (see the section below on symmetry of integrability) [3].

Arborescent iterations.— When we have $r>2$ involutions $I_1, I_2, \ldots, I_r$, the group Γ is of course still countable but gets exponentially big. Indeed elements of the group are "words" written with the letters I_k $(k=1,\ldots,r)$ with the restriction that consecutive letters have to differ. The number of words of a given length grows exponentially with this length, contrary to what happens when $r=2$.

The investigation of the orbits of Γ may be split into two steps. (1) The simplest is to study the subgroups generated by two arbitrarily chosen involutions I_{k_1} and I_{k_2}, in analogy with one-parameter subgroups of a continuous group, and this is how Fig. 1 was obtained. (2) The construction of the orbits under the full group may be approached by a random construction of typical elements of arbitrarily large length [4].

It is of course useful to produce admissible patterns. For simplicity we restrict ourselves here to $d=3$, and give two admissible patterns which generalize the $d=2$ pattern of the Baxter model. For both these patterns, $p=2$, M is an 8×8 matrix, and the indices take values ± 1. We define the patterns by restrictions on the entries of the matrix.

Pattern 1: Let us define M with the restrictions on the entries

$$M^{i_1i_2i_3}_{j_1j_2j_3} = M^{-i_1,-i_2,-i_3}_{-j_1,-j_2,-j_3}, \qquad (11)$$

$$M^{i_1i_2i_3}_{j_1j_2j_3} = 0 \text{ if } i_1i_2i_3j_1j_2j_3 = -1. \qquad (12)$$

These constraints amount to saying that the 8×8 matrix is the direct product of the same 4×4 matrix two times. It is further possible to impose that this matrix is symmetric, since such a symmetry is preserved by the partial transpositions t_1, t_2, t_3. Let us introduce the following notations for the entries of the 4×4 block of the matrix M:

$$\begin{pmatrix} a & d_1 & d_2 & d_3 \\ d_1 & b_1 & c_3 & c_2 \\ d_2 & c_3 & b_2 & c_1 \\ d_3 & c_2 & c_1 & b_3 \end{pmatrix}. \qquad (13)$$

The four rows and columns of this matrix correspond to the states $(+,+,+)$, $(+,-,-)$, $(-,+,-)$, and $(-,-,+)$ of the triplets (i_1,i_2,i_3) or (j_1,j_2,j_3). The matrix M can be completed by spin reversal, according to (11). Note that t_1 (respectively, t_2, t_3) simply exchanges c_2 with d_2 and c_3 with d_3 (respectively, circular permutations); I acts as the inversion of this 4×4 matrix.

Actually, and quite remarkably, there exist four quantities which are invariant by *all the four generating involutions, and therefore the whole group* Γ. Indeed consider

$$ab_1 + b_2b_3 - c_1^2 - d_1^2, \quad c_2d_2 - c_3d_3, \qquad (14)$$

and the polynomials obtained by permutations of 1, 2,

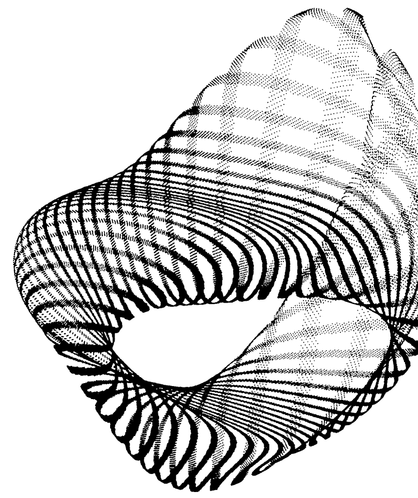

FIG. 1. The orbit of a generic point under the iteration of II_1 in $\mathbb{C}P_9$ (ten homogeneous parameters). We see that the orbit lies on a two-dimensional surface. There are indeed seven invariants common to I and I_1.

and 3. They form a five-dimensional space. Any ratio of the five independent polynomials is invariant under all the four generating involutions. In other words $\mathbb{C}P_9$ is foliated by five-dimensional algebraic varieties invariant under Γ.

Pattern 2: Let us define N by

$$N^{i_1i_2i_3}_{j_1j_2j_3} = f(i_1,i_2,i_3)\delta^{i_1}_{j_1}\delta^{i_2}_{j_2}\delta^{i_3}_{j_3} + g(i_1,i_2,i_3)\delta^{i_1}_{-j_1}\delta^{i_2}_{-j_2}\delta^{i_3}_{-j_3},$$

$$f(i_1,i_2,i_3) = f(-i_1,-i_2,-i_3), \qquad (15)$$

$$g(i_1,i_2,i_3) = g(-i_1,-i_2,-i_3). \qquad (16)$$

Equations (15) and (16) are symmetry conditions reducing the numbers of homogeneous parameters from 16 to 8. As for the previous model, there exists an invariant of the action of the whole group Γ:

$$\frac{f(+,+,+)f(+,-,-)f(-,+,-)f(-,-,+)}{g(+,+,+)g(+,-,-)g(-,+,-)g(-,-,+)}. \qquad (17)$$

For this model, the trajectories under II_1 are *curves* in $\mathbb{C}P_7$.

Symmetry of integrability.— One interest of the groups of transformations we described is that they enter *the symmetries of the Yang-Baxter equations* (also the star-triangle equations, tetrahedron equations, and the like). This symmetry accounts for the existence of the so-called spectral parameter, resolves the problem of the Baxterization [11], and gives a precious guideline for the solution of the equations [3].

The Yang-Baxter equation [10] for vertex models on a two-dimensional lattice is an algebraic constraint on the local Boltzmann weights:

$$\sum_{\alpha_1,\alpha_2,\alpha_3} R^{i_1i_2}_{\alpha_1\alpha_2}(1,2)R^{\alpha_1i_3}_{j_1,\alpha_3}(1,3)R^{\alpha_2\alpha_3}_{j_2j_3}(2,3) = \sum_{\beta_1,\beta_2,\beta_3} R^{i_2i_3}_{\beta_2\beta_3}(2,3)R^{i_1\beta_3}_{\beta_1,j_3}(1,3)R^{\beta_1\beta_2}_{j_1j_2}(1,2). \qquad (18)$$

If we set $A=tR(2,3)$, $B=\sigma t_1 R(1,3)$, $C=R(1,2)$ with σ defined as the exchange $(\sigma R)^{ij}_{kl}=R^{ji}_{kl}$, then the three involutions

$$K_A\colon A\to\sigma t I A,\ B\to t_1\sigma C,\ C\to\sigma t_1 B\,,$$

$$K_B\colon A\to\sigma t_1 C,\ B\to\sigma t I B,\ C\to t_1\sigma A\,,$$

$$K_C\colon A\to t_1\sigma B,\ B\to\sigma t_1 A,\ C\to\sigma t I C\,,$$

generate symmetries of the Yang-Baxter equations. It is easily verified that

$$(K_A K_B)^3=(K_B K_C)^3=(K_C K_A)^3=1\,. \qquad (19)$$

We get the affine Coxeter group $A_2^{(1)}$ described in [3]. This symmetry contains in particular cases the translation of the spectral parameter, related to the product IJ [1,4]. Such a shift often turns into a continuous translation, as would a rotation on the circle with irrational rotation number.

These results extend [4] to *higher-dimensional* avatars of the Yang-Baxter equations [12,13].

For example, when $d=3$ there are four involutions K_1,K_2,K_3,K_4 which satisfy various relations, for instance, $(K_1K_2K_3K_4)^2=1$. *The K_i's generate a group which is a symmetry group of the tetrahedron equations.* It is also a symmetry group for the three-dimensional vertex model *even if* the model does not satisfy the tetrahedron equation: For example, the partition function has definite automorphy properties under this group [2,14]. It provides an *extension to several complex variables functions of the notion of the fundamental group* $\mathbf{\Pi}_1$ of a Riemann surface, with, of course, a much more involved covering structure.

In conclusion, we have constructed a class of mappings and groups which clearly obey organizing principles. We believe there exists a classification of our mappings: It implies a classification of rational involutions of projective space *with integer coefficients,* and a description of the algebraic varieties they preserve.

They should be a good tool in the study of nonlinear systems in arbitrary dimensions, providing a definite example from chaotic to integrable behavior. They clearly will serve the study of the algebraic structure of integrable systems. (i) The finite-order orbits of our groups should be a mine of integrable models; see [1-4]. (ii) The "size" of the symmetry group in three dimensions should help disprove the existence of a "genuine" integrability in three dimensions [15], leaving room only for nontrivial dimensional reductions, or finite-order orbits. (iii) Our symmetry group is a group of automorphisms of the integrability varieties. This should give precious information on these varieties. In particular, one should decide if, up to Lie-group factors, these varieties can be anything else than Abelian varieties, or even product of curves. (For example, can they be K_3 surfaces?) (iv) An interesting point will be to exhibit the action of our symmetry group on the underlying quantum group for the Yang-Baxter equations [16,17].

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