Exactly Soluble Supersymmetric t-J-Type Model with Long-Range Exchange and Transfer

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The Gutzwiller wave function is shown to be the exact solution of a supersymmetric t-J-type model. The model realizes a Fermi-liquid state in one dimension with a discontinuity in the momentum distribution. Analytic results are obtained for spin and charge susceptibilities, and the specific-heat coefficient with the help of the Luttinger-liquid theory. In the high-density limit the model exhibits a Mott-Hubbard gap and reduces to an antiferromagnetic spin chain with long-range exchange solved by Haldane and Shastry.

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Exactly soluble one-dimensional fermion models such as the Tomonaga-Luttinger model [1], the Hubbard model [2,3], and the supersymmetric t-J model [4–7] show power-law singularities in the momentum distribution. This feature is in marked contrast to the discontinuity at the Fermi surface in Fermi liquids. In this Letter we present an interacting-fermion model that is exactly soluble and shows a discontinuity in the momentum distribution. This model is the first example that realizes a Fermi-liquid state with spin $\frac{1}{2}$ in one dimension. The model includes in the high-density limit the antiferromagnetic Heisenberg chain with long-range exchange which has been solved by Haldane [8] and Shastry [9]. We show that the Gutzwiller wave function is the exact solution of the model. The resultant Fermi-liquid state is identified as a free Luttinger liquid [1]. With this identification we obtain analytic results for most fundamental thermodynamic quantities such as the charge susceptibility, the spin susceptibility, and the low-temperature specific heat. The charge susceptibility indicates the presence of a Mott-Hubbard gap.

The t-J-type Hamiltonian is given by

$$\mathcal{H}=\mathcal{P}\sum_{i\neq j}\left[t_{ij}\sum_{\sigma}c_{i\sigma}^{\dagger}c_{j\sigma}+J_{ij}(\mathbf{S}_i\cdot\mathbf{S}_j-\tfrac{1}{4}n_in_j)\right]\mathcal{P}\,,$$

where $\mathcal{P}$ is the projection operator to exclude the double occupation at each site. Other notations are standard ones. We treat a finite system and impose the periodic boundary condition. Namely, we work with a ring of length N in units of the lattice spacing. We choose N even and require

$$t_{ij}=J_{ij}=tD(x_i-x_j)^{-2}\,,$$

with $D(x_i-x_j)=(N/\pi)\sin[\pi(x_i-x_j)/N]$ and $t>0$. In the macroscopic limit $-t$ reduces to the nearest-neighbor transfer.

The Gutzwiller state $|G\rangle$ is related to the free Fermi sea $|F\rangle$ by $|G\rangle=\mathcal{P}|F\rangle$. Following Ref. [10] we choose the fully polarized state $|N\uparrow\rangle$ of N electrons as the reference state and represent $|G\rangle$ as

$$|G\rangle=\sum_{\{x\},\{y\}}\Psi_G(\{x\},\{y\})\prod_{i\in\{x\}}c_{i\downarrow}^{\dagger}c_{i\uparrow}\prod_{j\in\{y\}}c_{j\uparrow}|N\uparrow\rangle\,.$$

Here $\{x\}$ denotes the set of coordinates for M down-spin electrons and $\{y\}$ denotes that of Q holes. Thus we have $N=2M+Q$. In order to remove the degeneracy we choose M odd. The amplitude $\Psi_G(\{x\},\{y\})$ is given, apart from a normalization factor, by

$$\Psi_G(\{x\},\{y\})=\exp\left[-i\pi\left(\sum_{\alpha}x_{\alpha}+\sum_{l}y_l\right)\right]\prod_{\alpha>\beta}D(x_{\alpha}-x_{\beta})^2\prod_{l>m}D(y_l-y_m)\prod_{\alpha,l}D(x_{\alpha}-y_l)\,. \tag{1}$$

This $\Psi_G(\{x\},\{y\})$ is a generalization of wave functions treated by Sutherland [11], who considered in the continuum space both a boson system and a fermion one, but not their mixtures. Under exchange of coordinates, holes behave as fermions and down-spin electrons behave as hard-core bosons which are called spin bosons hereafter. We note that the hard-core repulsion between holes and spin bosons leads to antisymmetry of $\Psi_G(\{x\},\{y\})$ under exchange of x_α and y_l. In this representation the system of spin bosons and holes has a total momentum of $-\pi(M+Q)$, which in fact is compensated by that of up spins constituting the vacuum.

The singlet nature of $|G\rangle$ appears as [10]

$$\sum_{x_1}\Psi_G(\{x\},\{y\})=0\,, \tag{2}$$

where x_1 is the first coordinate in $\{x\}$. In applying the transfer operator in $\mathcal{H}$ to $|G\rangle$ we consider the up-spin part $T_\uparrow$ and the down-spin one $T_\downarrow$ separately. $T_\uparrow$ changes $\{y\}$ in $\Psi_G(\{x\},\{y\})$ but leaves $\{x\}$ intact. On the other hand, $T_\downarrow$ causes exchange between pairs of x_α and y_l and is harder to treat. A crucial simplification occurs since $T_\uparrow|G\rangle$ is still a singlet. This follows because Eq. (2) is valid for any $\{y\}$ and thus after application of $T_\uparrow$ the summation over x_1 still gives zero for the wave function. By a rotation in spin space which does not change the singlet state, we can transform $T_\uparrow$ into $T_\downarrow$ and prove the equality $T_\uparrow|G\rangle=T_\downarrow|G\rangle$.

Let us rewrite the Hamiltonian in terms of the spin-boson operator b_i, where $b_i^{\dagger}b_i=n_i/2-S_i^z$, and the hole operator h_i defined by $n_i=1-h_i^\dagger h_i$. Using the singlet property described above we obtain

$$\mathcal{H}|G\rangle=\mathcal{P}\sum_{i,j}t_{ij}(2h_i^\dagger h_j+b_i^\dagger b_j+m_im_j-\tfrac{1}{2}n_in_j)|G\rangle\,, \quad (3)$$

with $m_i=b_i^\dagger b_i$. We represent the transfer operator for holes in Eq. (3) as T_h and that for spin bosons as T_b. Let $\Psi_{bG}(\{x\},\{y\})$ denote the coordinate representation of $T_b|G\rangle$, and $\Psi_{hG}(\{x\},\{y\})$ that of $T_h|G\rangle$. The ratio Ψ_{bG}/Ψ_G at $(\{x\},\{y\})$ is given by

$$\Psi_{bG}/\Psi_G=t\sum_{n=1}^{N-1}z^{-nN/2}D(n)^{-2}\sum_\alpha\prod_{\beta(\neq\alpha)}B_{\alpha\beta}^{(n)}\prod_l F_{\alpha l}^{(n)}\,, \quad (4)$$

where $z=\exp(2\pi i/N)$ and

$$B_{\alpha\beta}^{(n)}=1-[(1-z^n)Z_\alpha^2+(1-z^{-n})Z_\beta^2]/(Z_\alpha-Z_\beta)^2\,, \quad (5)$$

$$F_{\alpha l}^{(n)}=\cos(\pi n/N)+\sin(\pi n/N)\cot\Theta_{\alpha l}\,. \quad (6)$$

with $Z_\alpha=\exp(2\pi i x_\alpha/N)$ and $\Theta_{\alpha l}=\pi(x_\alpha-y_l)/N$. We use Greek indices for spin bosons and Latin ones for holes. The first term in $B_{\alpha\beta}^{(n)}$ and that in $F_{\alpha l}^{(n)}$ do not have coordinates of particles. With this in mind we expand Eq. (4) by the use of Eqs. (5) and (6) and classify terms according to the number of particles involved. Then we find that all terms with more than three particles vanish after summation over n. A similar observation has been made in Refs. [8] and [9] in the absence of holes.

For $\Psi_{hG}(\{x\},\{y\})/\Psi_G(\{x\},\{y\})$ we obtain

$$\Psi_{hG}/\Psi_G=2t\sum_{n=1}^{N-1}z^{-nN/2}D(n)^{-2}\sum_l\prod_m F_{lm}^{(n)}\prod_\alpha F_{l\alpha}^{(n)}\,. \quad (7)$$

Here again terms with more than three particles vanish. The three-body terms in Eqs. (4) and (7) consist of four types depending on the number of spin bosons involved. In the case of three spin bosons the three-body terms combine to a constant owing to the identity

$$\cot\Theta_{\alpha\beta}\cot\Theta_{\alpha\gamma}+\cot\Theta_{\beta\gamma}\cot\Theta_{\beta\alpha}+\cot\Theta_{\gamma\alpha}\cot\Theta_{\gamma\beta}=-1\,.$$

Similar reduction occurs in the case of three holes and that of two spin bosons and one hole.

In the case of two holes and one spin boson the three-body terms do not combine to a constant because of different numerical factors. However, by using the singlet property of Ψ_G we can transform the residual three-body term into

$$\sum_\alpha\sum_{l\neq m}\cot\Theta_{\alpha l}\cot\Theta_{\alpha m}\Psi_G=\left[\frac{1}{6}Q(Q-1)(Q-2-3N)+\sum_{l\neq m}\sin^{-2}\Theta_{lm}\right]\Psi_G\,.$$

The details of the algebra will be reported elsewhere.

We have thus seen that the sum of Eqs. (4) and (7) reduces to a constant plus two-body terms. The two-body terms turn out to be just equal to minus the interaction part in Eq. (3). This means that the Gutzwiller wave function $|G\rangle$ is an eigenstate of $\mathcal{H}$. The eigenvalue E is given in terms of $n_e=2M/N$ by

$$\frac{E}{N\pi^2t}=-\frac{n_e(n_e^2-3n_e+4)}{12}-\frac{1-2n_e/3}{N^2}\,, \quad (8)$$

By the nature of the method of solution it is hard to exclude the possibility of lower-energy states other than $|G\rangle$. Nevertheless there is strong evidence in favor of $|G\rangle$ being the ground state of the system. First, in the dilute limit, E given by Eq. (8) tends to that of the free Fermi sea. Thus if Ψ_G is not the ground state for finite n_e, a phase change should occur as the density is increased. This, however, is unlikely in view of the known properties of related models such as the Hubbard and supersymmetric t-J models. Second, in the high-density limit $n_e=1$, E agrees with the result of Refs. [8] and [9] with due account of the $-n_in_j/4$ term in $\mathcal{H}$. In this limit Haldane [8] has confirmed by exact diagonalization up to twelve sites that Ψ_G is indeed the ground state.

The charge susceptibility χ_c in the macroscopic limit is derived from Eq. (8) as

$$\frac{1}{\chi_c}=\frac{\partial^2(E/N)}{\partial n_e^2}=\frac{\pi^2t(1-n_e)}{2}\,.$$

The divergence of χ_c as n_e approaches 1 is consistent with the formation of the Mott-Hubbard gap, as in the Hubbard model [2] and the t-J model [7]. Let us compare χ_c with the susceptibility $\chi_c^{(0)}$ in the free model with the single-particle energy

$$\epsilon(k)=\pi t\{|k|(1-|k|/2\pi)-\pi(1-N^{-2})/3\}$$

for momentum k. The long range of t_{ij} makes its Fourier transform $\epsilon(k)$ dependent on the size of the system. We obtain $1/\chi_c^{(0)}=\pi^2t(1-n_e/2)/2$. The ratio

$$\tilde{\chi}_c=\frac{\chi_c}{\chi_c^{(0)}}=\frac{1-n_e/2}{1-n_e} \quad (9)$$

is a measure of the many-body effect. Interestingly the right-hand side of Eq. (9) agrees with the inverse of the discontinuity in the momentum distribution obtained by the Gutzwiller *approximation* for models with infinite repulsion.

We now show that the exact solution is consistent with the Luttinger-liquid theory [1]. For this purpose we shift for each spin σ the momentum distribution in the Slater determinant of $|G\rangle$ by $\pi J_\sigma/N$, with J_σ an even integer. Let us first consider the case where only the charge-current excitations are involved: $J_\uparrow=J_\downarrow$. Upon application of transfer operators in Eq. (3) to the shifted wave function, terms with more than three particles vanish as long as $|J_\uparrow|\leq M+1$, and the resultant state is shown to be an exact solution. We introduce the charge velocity v_c by $v_c=2/\pi\chi_c$ which reduces to the Fermi velocity in the noninteracting case. The increment of energy from that of Eq. (8) is calculated to be $\pi v_cJ_c^2/2N$, where the charge current J_c is defined [12] by $(J_\uparrow+J_\downarrow)/\sqrt{2}$. This result shows that v_c agrees with the charge-current velocity. The agreement leads to identification of the Fermi-liquid state $|G\rangle$ as a free Luttinger liquid [1]. Then the ratio of Eq. (9) also represents the enhancement factor of the effective mass for the charge current.

We next consider the case $J_\downarrow = 2J_\uparrow$. In this case both charge and spin excitations are involved. The resultant wave function is obtained by replacing in Eq. (1) the momentum $-\pi$ of each particle by $-\pi(1+K/N)$, with $K = J_\uparrow$. Although the state with $K \neq 0$ is not a singlet, close inspection shows that the transfer operator $T_\downarrow$ has the same effect as that of $T_\uparrow$. Therefore the same effective Hamiltonian as in Eq. (3) can be used. With the condition that $|J_\uparrow - J_\downarrow| \le 2$, terms with more than three particles vanish upon application of transfer operators in Eq. (3) and the shifted state is shown to be an eigenstate. The condition for the exact solution is rather strict in this case. We notice that this is a sufficient condition and suggest that the necessary condition is weaker.

The increment of energy for the case $J_\downarrow = 2J_\uparrow$ is calculated to be $K^2\pi^2 t(1-3n_e/4)/N$. By introducing the spin current [12] $J_s = (J_\uparrow - J_\downarrow)/\sqrt{2}$ and identifying the coefficient of J_s^2, we obtain the spin-current velocity v_s as

$$v_s = \pi t ,$$

which is independent of n_e. Using the property of the free Luttinger liquid we can derive the spin susceptibility χ_s from v_s: $\chi_s = 2/\pi v_s$. At $n_e = 1$ the result is consistent with that of Refs. [8] and [9] derived from the increment of the energy against changing the number M of spin bosons. We note that χ_s is smaller than the noninteracting one $\chi_s^{(0)}$ $(=\chi_c^{(0)})$. Namely, we have the ratio

$$\tilde{\chi}_s = \chi_s/\chi_s^{(0)} = 1 - n_e/2 .$$

The reduction of the homogeneous susceptibility is also present in the supersymmetric t-J model [7] and is due to the antiferromagnetic correlation.

The results for χ_c and χ_s are consistent with correlation-function exponents [13,14] for the Gutzwiller wave function, where no anomalous dimensions appear for either spin or charge. Namely, we have the exponents $K_\rho = K_\sigma = 1$ in the notation of Ref. [12]. We can derive the low-temperature specific heat with the aid of the formula obtained by the conformal field theory [3,7]. The specific-heat coefficient $\tilde{\gamma}$ normalized by the noninteracting one is given by

$$\tilde{\gamma} = \frac{\tilde{\chi}_c + \tilde{\chi}_s}{2} = \frac{(1-n_e/2)^2}{1-n_e} .$$

The many-body effect appears only at $O(n_e^2)$, in contrast to $\tilde{\chi}_c, \tilde{\chi}_s$ and the discontinuity $(1-n_e)^{1/2}$ of the momentum distribution [13] where the effect appears at $O(n_e)$. We note that $\tilde{\gamma}$ diverges as n_e approaches unity. The origin of divergence is the divergent density of states for charge excitations at the edge of the Mott-Hubbard gap [7]. Note that $\tilde{\chi}_c$ vanishes at exactly $n_e = 1$, and we obtain $\tilde{\chi}_s/\tilde{\gamma} = 2$ as in the half-filled case of the Hubbard model.

A peculiar feature of the present model is that the finite-size correction in Eq. (8) contains a nonuniversal contribution in addition to the universal one related to $v_c + v_s$, which is described by the conformal field theory [3,7]. This peculiarity comes from the size dependence in $\epsilon(k)$. Furthermore, the finite-size correction in Eq. (8) does not vanish in the dilute limit. This is not a problem since the exact solution in the form of Eq. (1) is valid only for odd M, which means that n_e has the minimum $2/N$. In fact, with $n_e = 2/N$ the energy is reduced to $2\epsilon(k=0)$, which is indeed exact as can be checked by solving the two-electron problem.

The reduction to the free fermion state in the dilute limit suggests a close relation to the supersymmetric t-J model [7]. In the latter model Yokoyama and Ogata [15] have observed by a numerical study that the Gutzwiller wave function is an excellent approximation not only for the ground-state energy but for structure factors and the momentum distribution at any density. However, a discrepancy appears in the exponent for correlation functions. It has been recognized [8,9] that in contrast to the Heisenberg model the long-range exchange model does not contain marginally irrelevant logarithmic corrections and represents the fixed-point model for the singlet spin liquid. The absence of logarithmic corrections holds for any density in the present supersymmetric model as can be seen in the explicit solutions for the correlation function for the Gutzwiller wave function [13,14]. In this sense the long-range supersymmetric model is regarded as a fixed-point model for Fermi liquids.

With slight modification of parameters in $\mathcal{H}$ the Fermi-liquid fixed point in one dimension should flow toward a Luttinger liquid with no discontinuity in the momentum distribution. The nature of the stable state depends sensitively on the direction of modification. If, for example, the parameters are such that the spin-dimer state is realized [8] in the high-density limit, introduction of holes may lead to a superconducting state. On the other hand, with perturbations such as interchain interactions which increase the dimensionality of the system, the Fermi-liquid fixed point should be greatly stabilized. The present model seems to be a useful reference model to study phase diagrams in the parameter space.

In conclusion, we emphasize that the simplicity of the Gutzwiller wave functions, suitably generalized to describe excited states as well, gives us a unique opportunity to study the behavior of the supersymmetric model without being restricted to the asymptotic regime. It has been pointed out in the high-density limit that there are enormous degeneracies in the excitation spectrum [8]. These degeneracies are called supermultiplets and interpreted in terms of "free spinons" [8]. In a preliminary study we have found in the two-electron system, which represents a dilute limit, that many singlet and triplet excited states are degenerate. Thus the present model poses further intriguing problems, such as whether the supermultiplet structures are present at any density, and what is the physical meaning of the degeneracy.

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[1] F. D. M. Haldane, J. Phys. C **14**, 2585 (1980); Phys. Rev. Lett. **45**, 1358 (1980).
[2] N. Kawakami and S.-N. Yang, Phys. Lett. A **148**, 359 (1990).
[3] H. Frahm and V. E. Korepin, Phys. Rev. B **42**, 10553 (1990).
[4] B. Sutherland, Phys. Rev. B **12**, 3795 (1975).
[5] P. Schlottman, Phys. Rev. B **36**, 5177 (1987).
[6] P.-A. Bares and G. Blatter, Phys. Rev. Lett. **64**, 2567 (1990).
[7] N. Kawakami and S.-N. Yang, Phys. Rev. Lett. **65**, 2309 (1990).
[8] F. D. M. Haldane, Phys. Rev. Lett. **60**, 635 (1988); **66**, 1529 (1991).
[9] B. S. Shastry, Phys. Rev. Lett. **60**, 639 (1988).
[10] P. W. Anderson, B. S. Shastry, and D. Hristpulos, Phys. Rev. B **40**, 8939 (1989).
[11] B. Sutherland, Phys. Rev. A **4**, 2019 (1971).
[12] H. J. Schulz, Phys. Rev. Lett. **64**, 2831 (1990).
[13] W. Metzner and D. Vollhardt, Phys. Rev. B **37**, 7382 (1988).
[14] F. Gebhard and D. Vollhardt, Phys. Rev. B **38**, 6911 (1988).
[15] H. Yokoyama and M. Ogata (to be published).