Comment on "New Exact Solution for the Exterior Gravitational Field of a Spinning Mass"

In a recent Letter,[1] Manko investigates the problem of describing the exterior field of a rotating mass. To solve this problem, one needs an exact solution of Einstein's equations containing the set of all "gravitoelectric" (due to mass) and "gravitomagnetic" (due to spin) multipole moments. In a series of publications,[2-7] we presented for the first time an exact solution satisfying these properties. The Kerr parameter is involved in the set of gravitomagnetic moments which, as physically expected, vanish if the angular momentum is set equal to zero. However, Manko claims that our solution "does not describe all possible deformations due to rotation." To describe the properties of the source, an infinite set of multipole moments is essentially all that can be invariantly defined based on an exterior solution; therefore, Manko's assertion does not have a sound foundation. Furthermore, Manko claims to have found a new general static axisymmetric asymptotically flat vacuum solution. I will show that (i) Manko's static solution is identical to the Weyl solution up to a coordinate transformation and a redefinition of the parameters entering the Weyl solution and (ii) Manko's stationary solution does not have any advantage over our work as regards deformations due to rotation.

To show (i), I consider the Weyl solution[8] in cylindrical coordinates (ρ,z), i.e.,

$$\psi=\frac{1}{2}\sum_{n=0}^{\infty}\alpha_n'\frac{P_n(\cos\theta)}{r^{n+1}},\quad r^2=\rho^2+z^2,\quad \cos\theta=\frac{z}{r}, \tag{1}$$

where α_n', $n=0,1,2,\ldots$, are constants and P_n represents the Legendre polynomial of order n. Equation (1) is a solution of the two-dimensional Laplace equation which is invariant under the translation $z\to z+\kappa$, corresponding to a constant displacement of the origin of coordinates along the symmetry axis. To write the Weyl solution in these "displaced" coordinates, we introduce prolate spheroidal coordinates x and y by $x=\kappa^{-1}(r_+ + r_-)/2$ and $y=\kappa^{-1}(r_+ - r_-)/2$, with $r_\pm^2=\rho^2+(z\pm\kappa)^2$. Moreover, we define parameters $\alpha_n=\alpha_n'-s_n$, where s_n correspond to the values of α_n' in Eq. (1) which lead to the Schwarzschild solution.[6] Accordingly, Eq. (1) becomes

$$\psi=\frac{1}{2}\ln\left(\frac{x-1}{x+1}\right)+\frac{1}{2}\sum_{n=1}^{\infty}\alpha_n(x+y)^{-n-1}P_n\left(\frac{xy+1}{x+y}\right). \tag{2}$$

This expression corresponds *exactly* to Manko's static solution, i.e., Eq. (9) of Ref. 1 with $f=\exp(2\psi)$.

By using the Hoensalaers-Kinnersley-Xanthopoulos (HKX) transformations,[9] Manko obtained the stationary generalization of the Weyl solution (2); however, this generalization was derived and discussed before by several authors.[10,11] Manko claims that this exterior solution for a rotating mass is more general than previous work[2-7] because Manko's metric contains two sets of arbitrary parameters: a set $\{\alpha_n\}$ that describes *static* deformations as well as a set $\{\beta_n\}$ that describes *stationary* deformations. This conclusion is erroneous since Manko's introduction of the "stationary" set occurs at the level of the static metric (2) by redefining the parameters α_n as $\alpha_n\to\alpha_n+q\beta_n$. Then the constant q is set equal to the parameter introduced by the HKX transformation which is the only one that can take the rotation of the source into account. That this is an artificial introduction of *stationary* parameters can also be seen from the fact that *all* gravitomagnetic multipole moments vanish in the limiting case $q=0$. The multipole moments in our general solution can be assumed to be functions of the parameters of the source; this fact is implicit in the *exterior* solutions presented thus far. The exact functional form for this dependence would follow from the smooth matching of our exterior solution to an appropriate interior solution along a boundary surface.

I would like to thank Professor Friedrich W. Hehl and Professor B. Mashhoon for helpful comments. This work was supported by the Deutsche Forschungsgemeinschaft, Bonn, Federal Republic of Germany.

Hernando Quevedo
Institute for Theoretical Physics
University of Cologne
D-5000 Cologne 41, Federal Republic of Germany

Received 15 May 1990
PACS numbers: 04.20.Jb

[1] V. S. Manko, Phys. Rev. Lett. **64**, 1625 (1990).

[2] H. Quevedo and B. Mashhoon, Phys. Lett. **109A**, 13 (1985).

[3] H. Quevedo, Phys. Rev. D **33**, 324 (1986).

[4] H. Quevedo, in *Proceedings of the Fourth Marcel Grossmann Meeting on General Relativity*, edited by R. Ruffini (North-Holland, Amsterdam, 1986).

[5] H. Quevedo, Ph.D. thesis, University of Cologne, Cologne, 1987 (unpublished).

[6] H. Quevedo, Phys. Rev. D **39**, 2904 (1989).

[7] H. Quevedo, Fortschr. Phys. (to be published).

[8] See, for example, D. Kramer, H. Stephani, E. Herlt, and M. MacCullum, *Exact Solutions of Einstein's Field Equations* (Deutscher Verlag der Wissenschaften, Berlin, 1980). The factor $\frac{1}{2}$ is conventional; it is used here in order to reach complete agreement with Manko's notations.

[9] *Solutions of Einstein's Equations: Techniques and Results*, edited by C. Hoenselaers and W. Dietz (Springer, Berlin, 1984).

[10] G. Neugebauer, J. Phys. A **13**, 1737 (1980).

[11] W. Dietz and C. Hoenselaers, Proc. Roy. Soc. London A **382**, 221 (1982).

