Roton-Vortex Interactions in Superfluid Helium

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From a new mathematical identity and by using the fractional difference between the momentum of a roton and the momentum at the bottom of the roton minimum as a small expansion parameter, a first-order perturbation treatment of the roton-vortex interaction is developed. The resulting analytic expressions for the distribution of transverse momentum transfer, in terms of elliptic integrals, are shown to be in excellent agreement with recent computer-simulation results of Samuels and Donnelly and are convenient for computing the total momentum transfer.

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Since the pioneering work of Hall and Vinen [1], the interaction of rotons with lines of quantized vorticity has received considerable attention. Because of the long range of the hydrodynamic vortex field, Lifshitz and Pitaevsky [2] noted that the motion of the rotons could be visualized in terms of wave packets and that consequently the transfer of momentum from a vortex line to the rotons could be calculated by a quasiclassical treatment of the roton trajectories. This method was subsequently employed by Goodman [3]. Following this same approach, Sonin [4] reported a detailed computation of the total momentum transfer by a summation over straight-line trajectories. Hillel [5] subsequently proposed a more direct computation by means of an integration over the hydrodynamic field itself, without the intermediary of the trajectories. Recently, Samuels and Donnelly [6] have presented the results of computer simulations of the actual quasiclassical roton trajectories, without explicitly using the straight-line approximation. Their Fig. 2, reproduced here as Fig. 1, shows the distribution of the fractional momentum transfer as a function of impact parameter b (in Å). Our purpose in this note is twofold: (1) to reexamine a shortcoming of the Hillel approach and to demonstrate by means of a *mathematical identity* how the Hillel method, when corrected, is equivalent to the Sonin summation over trajectories; and (2) to exhibit, by exploiting the existence of a small dimensionless parameter of the problem, closed analytic expressions for the transferred momentum distribution. The circles (Fig. 1) are computed from our analytic expressions, as to be discussed below. The good agreement serves to provide independent corroboration of the Samuels-Donnelly results and to confirm the reliability of the first-order perturbation approach.

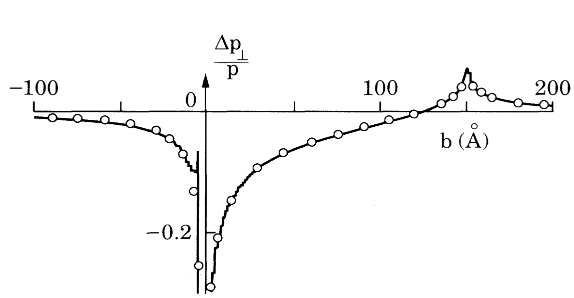

FIG. 1. Fractional transverse momentum transfer vs impact parameter (in angstroms) from the computer simulations of Samuels and Donnelly (Ref. [6]). The circles are calculated from the analytic expressions of Eqs. (12a)–(12c) for the characteristic impact parameter $b^* = 150$ Å and for the expansion parameter $\varepsilon = 0.033$.

From the standard Landau expression $H(p) = (p - p_0)^2/2\mu + \Delta$ for the momentum dependence of the roton energy near the minimum Δ, one obtains for the quasiclassical roton velocity,

$$V = \frac{\partial H}{\partial p} = \frac{p - p_0}{\mu}. \tag{1}$$

The vector velocity $\mathbf{V}$ is in the direction of the momentum vector $\mathbf{p}$. This velocity specifies the roton motion relative to the local superfluid flow, of velocity $\mathbf{u}$. The resultant roton velocity is $\mathbf{V} + \mathbf{u}$. By Galilean covariance, the local superfluid velocity shifts the roton energy by $U = \mathbf{p} \cdot \mathbf{u}$. Conservation of the roton energy E, measured relative to Δ, gives

$$E = \frac{\mu}{2} V_i^2 = \frac{\mu}{2} V^2 + U = \text{const}, \tag{2}$$

where we have employed Eq. (1) to eliminate the local momentum in terms of the velocity. The initial velocity of the roton, far from the vortex, is V_i, corresponding to the initial momentum p_i. By means of the constraint expressed by Eq. (2), we can regard the space dependence of V as providing the same information as that provided by the potential field itself. The local force on the roton is therefore $\mathbf{F} = -\text{grad}U = \mu V \text{grad} V$. Introducing the line element ds of the trajectory by the time derivative $ds/dt = V$, we find for the accretion of momentum in traversing ds, by substitution of the local force from above,

$$\frac{d\mathbf{p}}{ds} = \frac{dt}{ds}\frac{d\mathbf{p}}{dt} = \frac{1}{V}\mathbf{F} = \mu \,\text{grad} V. \tag{3}$$

This is the essential *mathematical identity,* evidently not hitherto employed in this problem, which connects the Sonin [4] and Hillel [5] approaches. In the former, the ratio of the spatial variations of **F** and V along a trajectory is integrated over s, yielding the momentum transfer for a given impact parameter b. Integration over b then yields the total transfer cross section. In computing this cross section, Hillel [5] sought to shorten the procedure by integrating the local force, $-\text{grad}U$, over the entire space, thereby obtaining a boundary integral over U. In this form, the Hillel approach is not justified (as apparently recognized by him) because of the exclusion of the excluded circular region of diameter

$$b^* = \frac{\mu\kappa p_i}{\pi(p_i - p_0)^2}, \tag{4}$$

where κ is the quantized circulation. This is the doubly cross-hatched region that is contained inside the contour labeled Γ_{ex} in Fig. 2, consisting of points where $U > E$. The other circles, labeled $\frac{1}{2}$, $\frac{1}{3}$, etc., are the potential-energy contours $U = E/2$, $E/3$, etc., respectively. Here we have introduced Cartesian coordinates with the vortex line at the origin and have used the approximation $U = \mathbf{p}\cdot\mathbf{u} \simeq p_i u_y$. The transverse momentum transfer $p_x(b)$ is calculated to first order in the small parameter $\varepsilon = (p_i - p_0)/p_i$ by integrating along the zero-order unperturbed straight-line trajectory parallel to the y axis at $x = b = \text{const}$. The dashed lines at 45° to the coordinate axes are the loci of points where $F_x = 0$. The lightly cross-hatched shadow region in Fig. 2 that occurs outside Γ_{ex} for $y > 0$ and $0 < b < b^*$ is compensated by the reflected or "snap-back" [3] trajectories of the lower

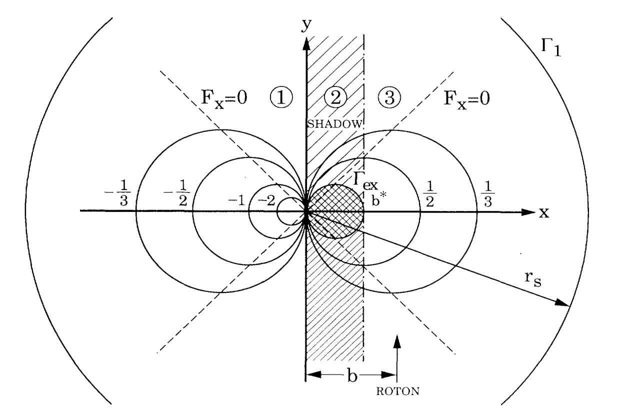

FIG. 2. Potential field set up by a vortex line at the origin for a roton of impact parameter b moving parallel to the y axis. The equipotentials are labeled by the ratio of the potential energy to the initial roton kinetic energy. The dashed lines labeled $F_x = 0$ are the loci of zero transverse force. Rotons with impact parameter in the range $0 < b < b^*$, with b^* determined by the initial kinetic energy, are excluded from the doubly cross-hatched circle inside Γ_{ex}, as well as from the shadow region. For a vortex lattice, the momentum transfer occurs within the unit cell bounded by Γ_1, of radius r_s.

more heavily cross-hatched region, so that the entire x-y plane is effectively filled up uniformly by straight-line trajectories, *except for the classically forbidden region inside* Γ_{ex}.

Although our ultimate interest is the interaction of a roton with a vortex lattice, it is convenient to consider first the scattering of a roton by an ideal isolated vortex line of zero core radius. This requires integrating over the unbounded interval $-\infty < y < \infty$, which yields

$$p_x^{sc}(b) = \mu \,\,⨍\!\!\!\!\times_{-\infty}^{\infty} dy \frac{\partial V}{\partial x}\bigg|_{x=b}, \tag{5}$$

where the "×" in the integral sign indicates omission of the excluded region $(x - b^*/2)^2 + y^2 < b^{*2}/4$. By Stokes's theorem, the total transfer is found to be

$$\int_{-\infty}^{\infty} p_x^{sc}(b)\,db = \mu \int_{-\infty}^{\infty} ⨍\!\!\!\!\times_{-\infty}^{\infty} dx\,dy \frac{\partial V}{\partial x} = \mu \oint_{\Gamma_0} dy\, V - \mu \oint_{\Gamma_{ex}} dy\, V. \tag{6}$$

In Hillel's version of the double integration, U would occur in the integrand instead of V. In our version, Γ_{ex} is the locus of the classical turning points, where $V = 0$, so that the last term in Eq. (6) vanishes identically. The remaining integration is carried out at large distances from the origin, over the contour Γ_0. This contour corresponds to the "slit" geometry of Hillel [5], for which the ratio of breadth in the x direction to that in the y direction is vanishingly small. Far from the origin, $U = p_i u_y \ll E$, which permits the expansion

$$V = \left[V_i^2 - \frac{2}{\mu}U\right]^{1/2} \simeq V_i - \frac{U}{\mu V_i} \simeq V_i - \frac{p_i}{p_i - p_0} u_y, \tag{7}$$

where we have substituted from Eq. (2). The closed path integral over the constant V_i vanishes, while the integration over u_y yields the quantized circulation κ, by virtue of the slit geometry. The total transfer cross section is therefore, in terms of the characteristic length of Eq. (4),

$$\int_{-\infty}^{\infty} p_x^{sc}(b)\,db = -\frac{\mu p_i}{p_i - p_0}\oint_{\Gamma_0} u_y\,dy = -\frac{\mu p_i}{p_i - p_0}\oint \mathbf{u}\cdot d\mathbf{l} = -\frac{\mu\kappa p_i}{p_i - p_0} = -\pi(p_i - p_0)b^*. \tag{8}$$

In terms of the dimensionless variables $\beta \equiv b/b^*$ and $\pi_x(\beta) \equiv \pi_x(b)/p_i$, Eq. (8) becomes

$$\int_{-\infty}^{\infty} \pi_x^{sc}(\beta)\,d\beta = -\pi\varepsilon. \tag{9}$$

Because U vanishes identically along the $\beta = 0$ trajectory passing through the origin, the total contribution of all of the negative impact parameters (region 1 of Fig. 2) is readily found to be

$$\int_{-\infty}^{0} \pi_x^{(1)}(\beta)\,d\beta = -\tfrac{1}{2}\pi\varepsilon, \tag{10a}$$

with an equal contribution coming from the integration over all of the positive impact parameters. By means of an elementary integration along the $\beta=1$ trajectory ($x=b^*$), one finds for the separate contributions of regions 2 and 3,

$$\int_0^1 \pi_x^{(2)}(\beta)d\beta=-2\varepsilon \tag{10b}$$

and

$$\int_1^\infty \pi_x^{(3)}(\beta)d\beta=(2-\tfrac{1}{2}\pi)\varepsilon . \tag{10c}$$

To find the full β dependence of the momentum transfer we taken the x differentiation in Eq. (5) outside the integral to obtain

$$\pi_x^{\rm sc}(\beta)=\frac{\partial}{\partial\beta}\beta \fint_{-\infty}^{\infty} d\eta\left[\left(1-\frac{\beta^{-1}}{1+\eta^2}\right)^{1/2}-1\right]. \tag{11}$$

The integration can be accomplished in terms of the complete elliptic integrals (details to be presented elsewhere [7]) yielding for the intervals $\beta<0$, $0<\beta<1$, and $1<\beta$,

$$\pi_x^{(1)}(\beta)=\varepsilon\frac{2\beta-1}{(\beta^2-\beta)^{1/2}}K\left(\frac{1}{\sqrt{1-\beta}}\right)+2\varepsilon(1-\beta^{-1})^{1/2}E\left(\frac{1}{\sqrt{1-\beta}}\right), \tag{12a}$$

$$\pi_x^{(2)}(\beta)=\varepsilon\beta^{-1/2}[K(\sqrt{\beta})-2E(\sqrt{\beta})], \tag{12b}$$

and

$$\pi_x^{(3)}(\beta)=\varepsilon(2-\beta^{-1})K\left(\frac{1}{\sqrt{\beta}}\right)-2\varepsilon E\left(\frac{1}{\sqrt{\beta}}\right), \tag{12c}$$

respectively. These functions are plotted versus β in Fig. 3 and are also exhibited by the circles in Fig. 1. The good agreement of the circles with the Samuels-Donnelly computer results [6] reproduced in that figure confirms the reliability of the first-order analytic treatment presented here.

Although noted by Hillel [5], it seems not to be generally appreciated that the integrated effect of the roton-vortex interaction is strongly dependent on the shape of the region of integration. This is because of the long range of the vortex velocity field, whose strength varies as $u=(\kappa/2\pi)(x^2+y^2)^{-1/2}$. The shape dependence of the line integral that occurs in Eq. (8) can be illustrated by studying the function

$$f(\gamma)\equiv\frac{1}{\kappa}\oint_{\Gamma_\gamma}u_y\,dy , \tag{13}$$

defined for an elliptical contour Γ_γ, where γ is the ratio of the x diameter to the y diameter. By an exchange of the coordinates we obtain

$$f(\gamma^{-1})=\frac{1}{\kappa}\oint_{\Gamma_{\gamma^{-1}}}u_y\,dy=\frac{1}{\kappa}\oint_{\Gamma_\gamma}u_x\,dx . \tag{14}$$

The sum of Eqs. (13) and (14) is

$$f(\gamma)+f(\gamma^{-1})=\frac{1}{\kappa}\oint_{\Gamma_\gamma}(u_x\,dx+u_y\,dy)=\frac{1}{\kappa}\oint\mathbf{u}\cdot dl=1 , \tag{15}$$

a functional relationship that requires $f(\gamma)-\frac{1}{2}$ to be an odd function of $\ln\gamma$. Actual computation for the elliptical contour gives $f(\gamma)=(1+\gamma)^{-1}$. The scattering was calculated for the slit region, for which $\gamma=0$ and $f(0)=1$. The momentum exchange between a roton and an individual vortex of a vortex lattice occurs in a symmetrical region, for which $\gamma=1$ and $f(1)=\frac{1}{2}$. The resulting integrated strength is

$$\int_{\rm sym}\pi_x(\beta)d\beta=-\tfrac{1}{2}\pi\varepsilon , \tag{16}$$

reduced by a factor of 2 relative to the scattering result of Eq. (9). The shape dependence introduces an additional impact-parameter dependence of the momentum transfer at the much larger length scale of the vortex lattice constant. The Wigner-Seitz type approximation of a circle of radius r_s for the unit cell leads to the difference

$$\pi_x^{\rm sym}(\beta)-\pi_x^{\rm sc}(\beta)=\varepsilon(b^*/r_s^2)(r_s^2-b^2)^{1/2}. \tag{17}$$

It is readily verified that integrating over Eq. (17) accounts for the difference between Eqs. (9) and (16).

The rotons that follow the reflected or snap-back [3] trajectories of region 2, for $0<\beta<1$, have final velocities in the negative y direction and final momenta reduced by $2(p_i-p_0)=2\varepsilon p_i$. Therefore the ratio of the x and y components of the transferred momentum is $\pi/4$.

We have used the present first-order approach (with details to be provided elsewhere [7]) to obtain analytic expressions for the trajectories themselves, in terms of incomplete elliptic integrals, employing the observation by Sanders [8] that angular momentum conservation in addition to energy conservation fixes the roton orbits by quadrature.

To summarize, we have obtained analytic expressions for the impact-parameter dependence of the roton momentum transfer resulting from the scattering by an

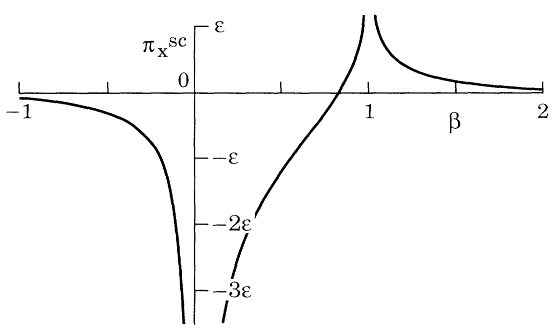

FIG. 3. Transverse momentum transfer $\pi_x^{\rm sc}$, in units of the initial momentum, vs $\beta=b/b^*$, the reduced impact parameter. The small perturbation parameter ε is the fraction by which the initial roton momentum differs from the momentum at the roton minimum.

ideal vortex line of zero core radius. The excellent agreement corroborates the recent computer-simulation result and confirms the reliability of the first-order perturbation method based on straight-line trajectories. A new mathematical identity yields directly the total transverse momentum cross section which is established to be $\pi/4$ times the total cross section for longitudinal momentum transfer.

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