Spectral Statistics in Semiclassical Random-Matrix Ensembles

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A novel random-matrix ensemble is introduced which mimics the global structure inherent in the Hamiltonian matrices of autonomous, ergodic systems. Changes in its parameters induce a transition between a Poisson and a Wigner distribution for the level spacings, $P(s)$. The intermediate distributions are uniquely determined by a single scaling variable. Semiclassical constraints force the ensemble to be in a regime with Wigner $P(s)$ for systems with more than two freedoms.

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The theory of random matrices has been used to describe various phenomena in condensed-matter physics, atomic physics, nuclear physics, and the semiclassical mechanics of nonintegrable systems. For example, these properly describe the conductivity fluctuations in mesoscopic systems[1] and the fine-scale spectral statistics of nuclei, atoms, molecules, and classically ergodic Hamiltonian models.[2] If time-reversal invariant, some spectral properties of such systems, e.g., the nearest-neighbor level-spacing distribution $P(s)$, resemble those of the Gaussian orthogonal ensemble (GOE) for a wide variety of examples. Actually, it has been shown that the GOE only represents a special case in a large family of ensembles which all have precisely the same spectral behavior.[3] In other words, the GOE spectrum is quite insensitive to modifications in the definition of the ensemble. Random matrices with properties similar to those of Hamiltonian matrices apparently form *another subset of the same large family* of ensembles as the GOE. However, the two subsets differ from each other in many ways. In particular, the global structure of semiclassical origin which is inherent to Hamiltonian matrices is absent in the GOE. While such differences are not reflected in the spectral properties, they will affect the properties of eigenstates which are not as robust.[3] In order to understand both the mechanism responsible for the robustness of spectral features and the properties of the eigenvectors, it is necessary to examine the behavior of random matrices which mimic the semiclassical structure of Hamiltonian matrices. Accordingly, the purpose of this Letter is twofold. First, we introduce an enlarged ensemble which includes both the traditional GOE and random matrices with Hamiltonian-like structure as subsets (different from that of Ref. 3) and study some of its properties. Specifically, we focus on checking whether or not the new ensemble is compatible with our expectation for the $P(s)$. For a particular ordering of the basis, the corresponding members of the new ensemble are banded. Thus, we refer to this model as the banded random-matrix ensemble (BRME). Its eigenstates are typically localized. Second, we show that in the Hamiltonian-like limit of the BRME its $P(s)$ is the same as that of the GOE and on average its eigenstates are extended. However, both of these conclusions hold only for systems with more than two freedoms, $d>2$. When $d=2$ our argument is marginal and a more detailed understanding of the relation between the BRME and the actual Hamiltonian matrices is required in order to determine the corresponding form of $P(s)$. Moreover, due to its additional structure, the BRME could lead to new predictions for various properties of ergodic Hamiltonian matrices which cannot be explained by the GOE.[4] The discussion of such predictions, however, will be postponed for a future publication.

Before we actually introduce the BRME, we give a short description of the semiclassical correlation structure of Hamiltonian matrices.[5] Suppose $\hat{H}=\hat{H}_0+\hat{H}_1$ and $\hat{H}_0\mathbf{v}_i=E_i\mathbf{v}_i$. We use $\mathbf{v}_i$ as the basis and arrange it in increasing order of the unperturbed energies E_i. In this basis, $\hat{H}$ is a banded matrix, $h_{ij}=E_i\delta_{ij}+\langle\mathbf{v}_i|\hat{H}_1|\mathbf{v}_j\rangle$. That is, matrix elements which lie much further away from the diagonal than a few bandwidths ΔE_i are vanishingly small, where

$$(\Delta E_i)^2\equiv\frac{\sum_j(E_i-E_j)^2|h_{ij}|^2}{\sum_{j(\neq i)}|h_{ij}|^2}\simeq\hbar^2\frac{\{[H_0,H]^2_{\rm PB}\}}{\{H^2\}-\{H\}^2}\,, \qquad (1)$$

and

$$\{F(q,p)\}\equiv\frac{\int dq\,dp\,F(q,p)\delta[E-H_0(q,p)]}{\int dq\,dp\,\delta[E-H_0(q,p)]} \qquad (2)$$

is a microcanonical average. Moreover, $[F,G]_{\rm PB}$ is the Poisson bracket and the $\simeq$ sign denotes equality to lowest order in $\hbar$, or, more precisely, the contribution from zero-length classical orbits.[6] Equation (1) is a consequence of the relation between microcanonical averages and diagonal elements, $h_{ii} \simeq \{H\}$. Let us now fix a classical range for E, (E_d, E_u), and truncate the basis $\mathbf{v}_i$ such that $E_d < E_i < E_u$. The resulting block spans a $O(\hbar^0)$ range in E while the band size ΔE_i is only $O(\hbar)$ [see Eq. (1)]. Accordingly, except for a thin band around the diagonal, all the elements of this block are vanishingly small. This observation can also be formulated in terms of numbers of rows and columns in the Hamiltonian matrix by using the Weyl formula for the density of states $\rho(E)$,

$$\rho(E) \simeq h^{-d} \int dq\, dp\, \delta[E - H_0(q,p)]\,. \qquad (3)$$

We obtain that the size of the block is $N \simeq \hbar^{-d}$ while the size of the band is smaller, $b \equiv \rho(E)\Delta E \simeq \hbar^{1-d}$. A second semiclassical constraint on the H matrix concerns its diagonal matrix elements. Since $h_{ii} \simeq \{H\}$, and due to the $\mathbf{v}_n$-basis ordering, these display on average a slow (global) variation with E.

We now introduce the banded random-matrix ensemble which, in contrast with the GOE, incorporates in a simplified form the global semiclassical structure of Hamiltonian matrices. The members of the GOE, $\mathcal{H}$, are $N \times N$ symmetric matrices with random, uncorrelated elements, $h_{ij} = G(0, \sigma + \sigma\delta_{ij})$, where $G(u,v)$ is a Gaussian distribution with mean u and variance v^2. Since σ only determines the overall size of the matrix elements, we set $\sigma = 1$. On the other hand, the matrices in BRME are in addition exactly banded, that is, $\langle h_{ij}^2 \rangle = 0$ whenever $|i-j| \geq b$. Moreover, their diagonal matrix elements have a mean which changes by α from one row to the next, $\langle h_{ij} \rangle = \alpha i \delta_{ij}$. For $b \geq N$ and $\alpha = 0$ the BRME becomes equivalent to the GOE. Similar ensembles were originally introduced by Wigner in the context of nuclear physics.[7] Recently, Deutsch introduced the BRME as an example of a closed quantum system with a large number of freedoms which is ergodic.[8] Moreover, such matrices also arise in the study of tight-binding models with an external electric field for an electron on a disordered 1D lattice.[9]

In order to further stress the analogy to Hamiltonian matrices, we define the enlarged BRME. The latter includes all the matrices obtained from the BRME by permutations of the basis in which the BRME itself is expressed. As in the case of a generic Hamiltonian matrix the nonlocal correlation structure is hidden for most members of the enlarged BRME. On the other hand, the spectral properties of the enlarged BRME are precisely the same as those of the BRME itself. We therefore can restrict our study to the BRME without loss of generality.

If $N = \infty$ and b is finite, the eigenvalue equation for a member of the BRME can be cast in a transfer-matrix form. Consequently, it falls under the auspices of the Furstenberg theorem which implies that the corresponding eigenvectors are exponentially localized.[10] Another limit of the BRME, where N is finite and $\alpha = 0$, was recently studied by Casati *et al.*[11] They found that $L/N = f(b^2/N)$, where L is the average localization length. The *scaling function* $f(x) = Cx$ for small x where $C \approx 1$ and saturates to 1 when x is large.

The most extensively studied spectral property of random matrices is $P(s)$, the distribution of spacings between consecutive eigenvalues S, where $s = S\rho(E)$. For both the GOE and ergodic Hamiltonians, $P(s)$ is very well approximated by the Wigner distribution, $P(s) = (\pi s/2)\exp(-\pi s^2/4)$. The fact that $P(0) = 0$ is a signature of the repulsion between levels. On the other hand, integrable systems with $d > 1$ have been shown to display no level repulsion; $P(s) = \exp(-s)$, the Poisson distribution.[12] For the BRME, if $N = \infty$ and $\alpha = 0$, the localization of eigenstates implies that the overwhelming majority of eigenvalues have negligible repulsion. As a consequence, the spacings are Poisson distributed. For the ensemble studied in Ref. 11, a transition from a Poisson to a Wigner distribution was observed as the appropriate scaling variable, $x \equiv b^2/N$, was gradually increased. In the following, we show that by varying α such a transition is also obtained for $N = \infty$.

We now turn to the study of the BRME with finite α but very large N. In particular, we attempt to quantitatively understand the behavior of its spacing distribution. It is natural to assume that the local $P(s)$, namely, that restricted to eigenvalues with eigenvectors which are localized within L sites of each other, is of Wigner type. On average, such eigenvalues correspond to diagonal elements which are located within L rows away of each other. Moreover, this local spectrum lies within some energy interval $(\mathcal{E}_d, \mathcal{E}_u)$ of width $\Delta\mathcal{E}$. When $\alpha = 0$, the spectrum of the BRME results from incoherently overlaying a large number of local spectra and this leads to a Poisson spacing distribution. On the other hand, if α is finite, the intervals $(\mathcal{E}_d, \mathcal{E}_u)$ associated with individual local spectra are shifted with respect to each other along the energy axis. In particular, if $L\alpha > \Delta\mathcal{E}$, these intervals do not overlap at all. In this case, the spacings of one local spectrum are not altered by intervening eigenvalues from other local spectra and therefore the Wigner distribution $P(s)$ is preserved in the full BRME. In order to characterize the intermediate situations where $0 < L\alpha < \Delta\mathcal{E}$, we define a new scaling variable $\gamma \equiv L\alpha/\Delta\mathcal{E}$, which measures the relative strength of the two mechanisms causing the spread in energy: (1) the $\alpha = 0$ natural width of the local spectrum, and (2) the amount of α shift from one local spectrum to the next. The central assumption of our description is that these two mechanisms *do not interfere* with each other.[13] Accordingly, we assume that both L and $\Delta\mathcal{E}$ are independent of α; $L = b^2$,[11] and $\Delta\mathcal{E} \approx \sqrt{b}$ (see next paragraph for derivation).[14] Thus, $\gamma \approx \alpha b^{3/2}$. As either α or b grows, the spacing distribution displays a gradual transition between a Poisson and a Wigner form. In the process of this transition the intermediate forms of the spacing distribution are uniquely determined by the value of γ.

In order to gain additional insight into the behavior of $P_\gamma(s)$, we can further define the nature of the local spectra. This is achieved by approximating the band of the BRME with $L\times L$ blocks centered on the diagonal such that the upper left corner of one block lies on the diagonal of the matrix and is adjacent to the lower right corner of the next block. While $\alpha=0$ inside each block, the average of the diagonal elements differs by $L\alpha$ from one block to the next. In the following, we refer to this model as the block ensemble. Moreover, we assume that the spacing distribution for each of the blocks is Wigner type and the corresponding density of states is in the form of a semicircle (as in the GOE). Since $\langle\lambda^2\rangle=N^{-1}\times\langle\mathrm{Tr}\mathcal{H}^2\rangle$ (λ are the eigenvalues of $\mathcal{H}$), an exact calculation of the semicircle width only implies counting the nonvanishing matrix elements of $\mathcal{H}$. For $1\ll b\ll N$, $\Delta\mathcal{E}=4\sqrt{2b}$ and therefore $\gamma=\frac{1}{2}\alpha(b/2)^{3/2}$. The local densities of states form a periodic 1D lattice of partially overlapping semicircles (the lattice constant is $L\alpha$). One can use the approach of Gurevich[15] and Pevsner to derive the theoretical $P_\gamma(s)$ for the block ensemble. Despite the various simplifications, the theoretical $P_\gamma(s)$ displays nice qualitative agreement with the $P_\gamma(s)$ of the BRME.

For an independent one-parameter characterization of the intermediate forms of $P(s)$ we use the Brody distribution, $P_q(s)=\beta s^q\exp(-\kappa s^{1+q})$, where $\beta=(1+q)\kappa$ and $\kappa=\Gamma^{1+q}((2+q)/(1+q))$. While at $q=0$ this gives a Poisson distribution, for $q=1$ it is Wigner. The Brody formula was derived assuming an s^q repulsion between adjacent levels.[16] In Fig. 1 we numerically test the validity of the one-variable scaling description for the spacing distribution. We fit the Brody distribution to that obtained by numerically diagonalizing 125 BRME matrices with $N=800$ (see Fig. 2). To reduce finite-size effects, eigenvalues corresponding to eigenvectors localized less than L sites from either end are not included. For $\gamma>0.3$, q clearly scales with γ, $q(\gamma)$. While in the $N=\infty$ limit, $q(0)=0$, numerically we are faced with finite-size effects. These lead to $q(0)>0$ which in turn is a consequence of having only a finite number of blocks in each matrix. Moreover, one can easily show that the finite-size effects start at $\gamma\equiv\gamma_{\mathrm{cr}}\approx b^2$ where $\Delta\mathcal{E}$ becomes of the same order as the energy spread of the entire matrix, $N\alpha$. Such effects can be accounted for with a two-variable scaling function $q(\tilde{x},\gamma)$, where $\tilde{x}=L/N$. Keeping $\tilde{x}$ fixed is equivalent to having a constant number of blocks in each matrix. Notice that, in Fig. 1, $\tilde{x}$ varies. Numerical experiments in which $\tilde{x}$ was kept fixed ($b=10$, $N=556$) were also performed. It was found that the resulting $q(\gamma)$ curve precisely overlaps with the $b=12$ curve of Fig. 1 and does so equally well for all values of γ. We should point out that the scaling behavior of q is more robust than it might appear from our discussion. In particular, we have shown that $L=b^2g(y)$, where $y=\alpha b^{3/2}$, and $g(y)=C_0$ for $y\ll 1$ and $g(y)=C_1\times y^{-2/3}$ when $y\gg 1$.[17] This implies that $\gamma\approx yg(y)$ and therefore the assumption that L does not depend on α is unnecessary.

Finally, we discuss the implications of the BRME properties to autonomous, ergodic Hamiltonian systems. Using Eqs. (1)-(3) and $\sigma=\simeq\hbar^{(d-1)/2}$,[18] we obtain that $y\simeq\hbar^{2-d}$. Notice that, $y(\sigma)=y(1)\sigma^{-1}$. Accordingly, in the limit $\hbar\to 0$ and for $d>2$, y is diverging. In the large-y regime, $\gamma\approx y^{1/3}$ and so $\gamma\to\infty$. Therefore, the semiclassically constrained BRME agrees with the GOE with respect to the form of the spacing distribution. Namely, both predict a Wigner-type $P(s)$. As a matter of fact, the semiclassical ergodic Hamiltonians are even further away from the Poisson-Wigner transition than it might appear from the previous argument. In the study of the BRME we have implicitly assumed that the finite-size scaling variable $\tilde{x}$ is small. Semiclassically,

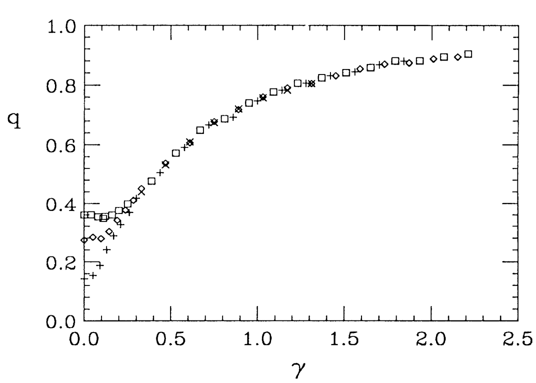

FIG. 1. The scaling hypothesis $q(\gamma)$. The data points are obtained from ensembles of 125 matrices with $N=800$ and correspond to different bandwidths: $b=8$ (+), $b=10$ (×), $b=12$ (◇), $b=14$ (□).

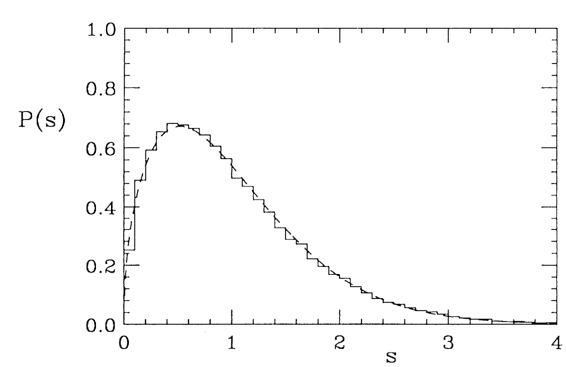

FIG. 2. The $P(s)$ distribution for one of the points in Fig. 1 (histogram) compared with the best-fitting Brody distribution (dashed line) ($\gamma=0.4$, $b=10$, $q=0.484$).

however, $\tilde{x} \simeq \hbar^{(2-d)/3}$ as well, and consequently $L \to N$ when $\hbar \to 0$ and $d > 2$. Thus, to the extent to which the properties of the BRME coincide with those of Hamiltonian matrices, localization is irrelevant for $d > 2$ and $\hbar \to 0$. While the GOE leads to the same conclusion, we stress that for the BRME this semiclassical limit is attained in a nontrivial way which one should be able to observe in actual Hamiltonian systems.

The $d=2$ case is semiclassically marginal and will require further study. Let us refer to $(\Delta E)^2$ of Eq. (1) as the second moment of h_{ij}. All the higher moments are also constrained by semiclassical expressions analogous to that of Eq. (1).[18] These can be thought of as additional correlations which are not included in the BRME. We expect that these will further enhance the localization length in the case of Hamiltonian matrices such that semiclassically $L \approx N$ and $\gamma \gg 1$ also when $d=2$.

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