Fractal Dimension in Nonhyperbolic Chaotic Scattering

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In chaotic scattering there is a Cantor set of input-variable values of zero Lebesgue measure (i.e., zero total length) on which the scattering function is singular. For cases where the dynamics leading to chaotic scattering is nonhyperbolic (e.g., there are Kolmogorov-Arnol'd-Moser tori), the nature of this singular set is fundamentally different from that in the hyperbolic case. In particular, for the nonhyperbolic case, although the singular set has zero total length, we present strong evidence that its fractal dimension is 1.

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In a typical scattering problem, one considers particles incident from infinity which interact with a localized potential (the scatterer) and then move off to infinity. It has recently become clear that, in many problems of this type, the basic dynamics can be chaotic, and the associated phenomenon has been called chaotic scattering.[1] A defining attribute of chaotic scattering is the singular behavior exhibited by scattering functions. By a scattering function we mean a plot of an output variable characterizing the trajectory after scattering versus an input variable characterizing the incident trajectory (e.g., scattering angle versus impact parameter). In chaotic scattering the scattering function is singular on a Cantor set of values of the input variable.[1,2] In any *arbitrarily small* neighborhood of such a singular input-variable value, the output variable varies wildly, and the range of variation of the output variable does not decrease to zero as the size of such a singular input-variable neighborhood is reduced. Numerical experiments demonstrating this striking behavior have been performed.[1-3] Furthermore, it has been shown that the singular set corresponds to initial conditions which yield orbits which enter the scattering region but never leave it.[1-3] The existence of this type of behavior of the scattering function means that relatively small uncertainty in the input variable can often make determination of the output variable impossible. A quantitative measure characterizing the magnitude of this effect is the fractal dimension of the set of singular input-variable values. For the case where the chaotic scattering set is *hyperbolic,* this dimension is typically less than 1 and greater than 0.[2,3] It is the purpose of this Letter to consider the fractal set of singular input-variable values for the case where the chaotic scattering set is *nonhyperbolic.*[4,5] In this Letter we do not give a precise definition of hyperbolic dynamics. We do note, however, that when the dynamics is hyperbolic, all periodic orbits are unstable, and there are no Kolmogorov-Arnol'd-Moser (KAM) tori. Nonhyperbolic chaotic scattering is typically characterized by the presence of KAM tori in the scattering region. Our main result is that in the nonhyperbolic case, although the singular set has zero total length (zero Lebesgue measure), its fractal dimension is always 1. This means that the difficulty of determining outputs from inputs having small uncertainty is (in a sense to be discussed) maximal for nonhyperbolic chaotic scattering.

It is instructive at this point to consider a simple example of a zero-Lebesgue-measure Cantor set which has fractal dimension $d=1$. The set is constructed in stages as follows. Start with the interval [0,1]. Remove the open middle third interval. From each of the two remaining intervals remove the middle fourth interval. Then from each of the four remaining intervals remove the middle fifth, and so on. At the nth stage of the construction, there will be $N=2^n$ intervals, each of length $\epsilon_n = 2^{-n}[2/(n+2)]$. The total length of all intervals $\epsilon_n N \sim n^{-1}$ goes to zero as n goes to infinity. For a covering of the set by the ϵ_n intervals we have $N(\epsilon) \sim \epsilon^{-1} \times (\ln\epsilon^{-1})^{-1}$. The box-counting (or "capacity") dimension is $d = \lim_{\epsilon\to 0}[\ln N(\epsilon)]/(\ln\epsilon^{-1})$, and clearly yields 1. Basically, d is the exponent of the dependence $N(\epsilon) \sim 1/\epsilon^d$, where the weaker logarithmic dependence is immaterial in the determination of the dimension. We note, however, that it is the logarithmic term which is responsible for ensuring that the Lebesgue measure is zero: $\epsilon N(\epsilon) \sim (\ln\epsilon^{-1})^{-1}$ approaches 0 as $\epsilon \to 0$.

To generalize this example, if at each stage we remove a fraction $\eta_n = \alpha/(n+c)$ (where α and c are constants) from the middle of each of the 2^n remaining intervals, then we find that

$$N \sim \frac{1}{\epsilon}\left(\ln\frac{1}{\epsilon}\right)^{-\alpha}. \qquad (1)$$

We note from (1) that the slope of the curve $\ln N$ vs $\ln(\epsilon^{-1})$, $d(\ln N)/d(\ln\epsilon^{-1})$, is always less than 1 for small nonzero ϵ, and it approaches 1 slowly (i.e., logarithmically) as $\epsilon \to 0$ (again yielding $d=1$). Thus, for fractals whose general character is similar to that for this example, we expect that accurate numerical estimation of the dimension will require going to very small scales. In addition, we also expect that any numerical estimation of the dimension made using a finite range of scales will be an underestimate, but that as the scale is made much smaller the numerically determined value will increase toward 1.

The reason why we suspect that the dimension of the fractal set of singularities is 1 in the nonhyperbolic case is that in nonhyperbolic situations there is an algebraic decay with time of the fraction R of incident particles that remain in the scattering region,[6] $R(t) \sim t^{-\alpha}$. This algebraic decay is in contrast with the exponential decay[1,2,3(b)] found in the hyperbolic case, $R(t) \sim \exp(-\gamma t)$. Based on previous analyses of the dynamics of chaotic scattering,[1,2,3(b)] we can imagine that, in the hyperbolic case, a standard (i.e., $d<1$) Cantor set results qualitatively as follows.[7] There is an interval of input variables which lead to trajectories that remain in the region of the scatterer for at least a duration of time T_0. By time $2T_0$ a fraction η of these particles leave. Say that the initial conditions of these escaping particles are all located in the middle of the original interval. Then we are left with two equal-length intervals of the input variable which yield trajectories which remain for at least a time $2T_0$. By time $3T_0$ say that an additional fraction η of the particles remaining at time $2T_0$ escape. Assuming that they do so from the middles of the two intervals at time $2T_0$, we now have four remaining intervals. Proceeding in this way, we obtain a Cantor set of dimension $d = \ln 2/\ln[(1-\eta)/2]^{-1}$ on which particles never escape. The exponential decay of the remaining particles with time, $R(t) \sim \exp(-\gamma t)$, in this case has a decay rate $\gamma = T_0^{-1}\ln(1-\eta)^{-1}$. This simple picture captures the essence of what happens in the hyperbolic case.[2,3(b)] As a crude indication of the behavior in the *nonhyperbolic* case, we can imagine following the same process, but with algebraic (rather than exponential) decay. Then the fraction of particles which leave at each stage η_n is no longer constant, but decreases with n. In particular, $\eta_n \cong -T_0 R^{-1} dR/dt$ for large n, giving $\eta_n \cong \alpha/n$. This yields Eq. (1) and hence a dimension-one Cantor set. [Thus for this model the power α of the logarithmic term in (1) is identified with the algebraic-decay exponent.] Clearly, the above is only a heuristic argument which leads to the conjecture that the dimension might be 1 in nonhyperbolic cases. In what follows we present numerical evidence that strongly supports this conjecture.

To facilitate our numerical calculations, we use an area-preserving map rather than a continuous-time Hamiltonian system,

$$
\begin{aligned}
x' &= \lambda[x-(x+y)^2/4]\,,\\
y' &= (1/\lambda)[y+(x+y)^2/4]\,,
\end{aligned}
\tag{2}
$$

where we take the parameter λ to be greater than 1. This quadratic map[8] has one fixed point at the origin (unstable) and another $x=\lambda y$, $y=4(\lambda-1)/(\lambda+1)^2$ (stable for $1<\lambda<3+\sqrt{8}=5.828$ and unstable for $\lambda>3+\sqrt{8}$). For large x and y the quadratic terms in (2) dominate, and x quickly moves to $-\infty$. In addition, it can be shown that any initial condition in $x<0$ moves to $x=-\infty$. Thus any invariant set of (2) must lie in $x \geq 0$ and in a finite region $(x^2+y^2<r^2)$ about the origin. We have numerically examined (2) in a range of the parameter λ, $1 \leq \lambda \leq 15$, and we find that the invariant set is apparently hyperbolic (nonhyperbolic) for $\lambda \gtrsim 6.5$ $(\lambda \lesssim 6.5)$. That is, we observe no KAM tori and the decay is exponential for $\lambda \gtrsim 6.5$.

We define the time delay T for an initial point (x_0, y_0) as the number of iterations it takes to reach $x<x_f<0$. Specifically, we choose $y_0=-3$ and $x_f=-3$ and consider T as a function of x_0. The function $T(x_0)$ for a typical nonhyperbolic case is shown in Fig. 1(a) for $\lambda=4.1$. Figure 1(b) shows a blowup of $T(x_0)$ in a small

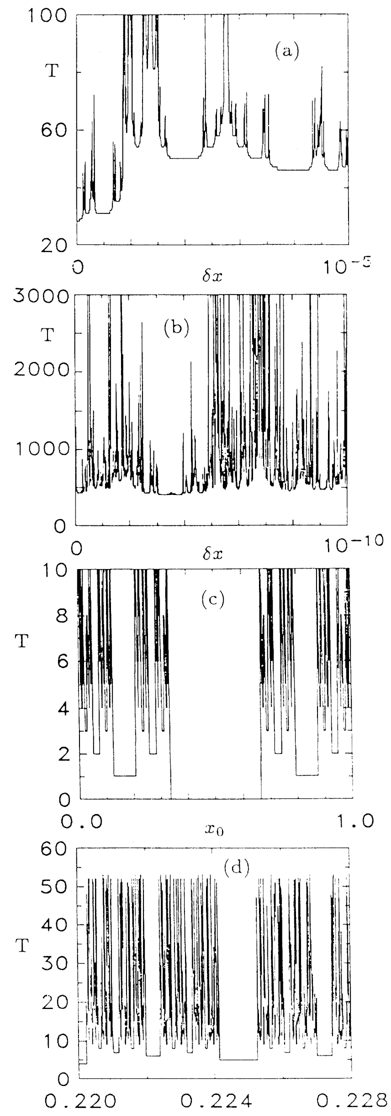

FIG. 1. Time-delay functions $T(x_0)$. (a) For the quadratic map with $\lambda=4.1$ with $x_0=1.03302+\delta x$ and δx in the range $0 \leq \delta x \leq 10^{-5}$. (b) Same as (a) but with $x_0=1.03302270685+\delta x$ and δx in the range $0 \leq \delta x \leq 10^{-10}$. (c) For the $d=1$ Cantor set with $\eta_n = 1/(n+2)$. (d) An expanded plot of (c).

interval where Fig. 1(a) indicates a long time delay. The horizontal axes in Figs. 1(a) and 1(b) are $\delta x = x_0 - x_{a,b}$, where $x_a = 1.03302$ and $x_b = 1.033022706 85$, respectively. For comparison, Fig. 1(c) shows the time-delay function resulting from the model dimension-one Cantor set described previously, while Fig. 1(d) shows a blowup of a small interval in Fig. 1(c). The relevant point is that the density of singularities appears to be much greater in the blowups than in the plots over the larger intervals, and this is true for the result both from Eq. (2) and from the dimension-one Cantor set model. This is *qualitatively different* from what one observes in the hyperbolic case. In particular, in the hyperbolic case the general character of the behavior of $T(x_0)$ in blowups is qualitatively indistinguishable from that on larger scales.[3] The reason for the denser appearance of the singularities in the blowup for the case of the one-dimensional Cantor-set model is that the successively removed intervals get relatively smaller as the delay time increases (i.e., η_n decreases with n).

In measuring the dimension of the Cantor set in the nonhyperbolic case, it is essential, as previously discussed, to use small enough ϵ. Estimation of the usual box-counting dimension requires an extremely large number of intervals when ϵ is very small (like 10^{-10}). Computation of the *uncertainty dimension*[3(b),9] is much less demanding, and this is the dimension we shall use. The uncertainty dimension for the time-delay function is calculated as follows. For a fixed value of "uncertainty" ϵ, we randomly choose a point x_0 and compute $|T(x_0) - T(x_0+\epsilon)|$. If $|T(x_0) - T(x_0+\epsilon)| > 0.5$, then we say that x_0 is ϵ uncertain. (The subsequent results for the dimension are independent of the choice 0.5 in the inequality.) We then randomly choose another x_0 and decide whether it is uncertain. We continue doing this until the number of uncertain points reaches a prescribed figure (typically 100). We then divide the number of uncertain points by the total number of points chosen (both certain and uncertain) to obtain the uncertain fraction $f(\epsilon)$. The uncertainty dimension is defined as $d = 1 - \lim_{\epsilon \to 0}[\ln \bar{f}(\epsilon)/\ln \epsilon]$, where $\bar{f}(\epsilon)$ is the probability that a randomly chosen x_0 is uncertain, and is numerically approximated by $f(\epsilon)$. After calculating $f(\epsilon)$ for many values of ϵ, we plot $f(\epsilon)/\epsilon$ as a function of ϵ on a log-log scale. Typically we observe that such plots are well fitted by a straight line, indicating that the uncertain fraction scales as a power of ϵ, $f(\epsilon) \sim \epsilon^\beta$, which gives an estimate for the uncertainty dimension, $d = 1 - \beta$. Thus the uncertainty dimension essentially tells us how the probability of making an error of order 1 or more (e.g., greater than 0.5 in our computations) in determining $T(x_0)$ for randomly chosen x_0 scales with uncertainty ϵ in x_0. When d is larger this scaling is more unfavorable (i.e., the degree to which one is able to improve one's predictive power [reduce $f(\epsilon)$] by decreasing the uncertainty ϵ is more limited), and the most unfavorable scaling possible is $d = 1$ (since $d \leq 1$). When d is close to 1, computation of the uncertainty dimension is especially effective because $f(\epsilon)$ is large and the total number of required points is relatively small. From the definition of the uncertainty dimension it can be shown that it is smaller than or equal to the box-counting dimension.[9] The uncertainty and box-counting dimensions have been conjectured to be equal in typical dynamical systems,[9] and classes of systems for which rigorous results can be obtained satisfy this.[10] Furthermore, we can prove that our model $d = 1$ Cantor set also has an uncertainty dimension of 1, in agreement with its box-counting dimension.

Figure 2(a) (for $\lambda = 4.1$) shows results for $\log_{10}(f/\epsilon)$ vs $\log_{10}\epsilon$ using points x_0 randomly chosen in the interval $x_a \leq x_0 \leq x_a + \Delta x$, where $\Delta x = 10^{-5}$ [$T(x_0)$ in this range is plotted in Fig. 1(a)]. The slope of the fitted line gives $d \cong 0.79$. Figure 2(b) shows similar results, but now using the much narrower range of x_0 shown in Fig. 1(b), $x_b \leq x_0 \leq x_b + \Delta x$, where $\Delta x = 10^{-10}$. The slope in this case gives a value of d significantly closer to 1 ($d \cong 0.96$). One reason for employing this zoom-in technique is that there may be large intervals where the discernibility of the possible one-dimensional nature of the set requires much more magnification than in other intervals[11] (e.g., see Figs. 16 and 18 of Ref. 5). The zoom-in technique allows us to focus on those intervals where the higher-dimensional structure is more easily measured. In addition, the use of very small ϵ is expected to be necessary in order to obtain dimensions close to 1. In particular, in our model the reason the dimension is 1 is that the removed fractions η_n eventually get very small,

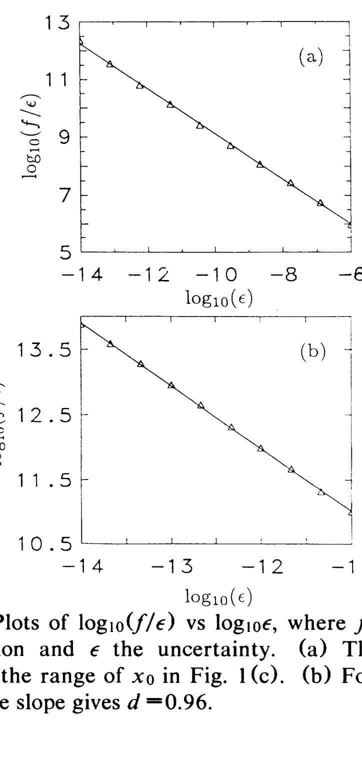

FIG. 2. Plots of $\log_{10}(f/\epsilon)$ vs $\log_{10}\epsilon$, where f is the uncertainty fraction and ϵ the uncertainty. (a) The slope gives $d = 0.79$ for the range of x_0 in Fig. 1(c). (b) For the range in Fig. 1(d), the slope gives $d = 0.96$.

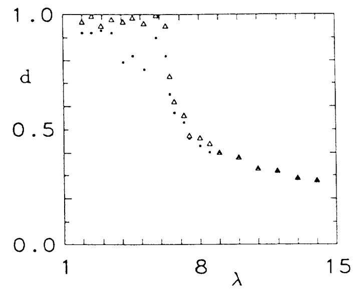

FIG. 3. The uncertainty dimension for the quadratic map vs λ.

and this only happens at sufficiently small scale. This can also be clearly seen by applying the uncertainty-dimension technique to the model $T(x_0)$ shown in Figs. 1(c) and 1(d). The result for the range shown in Fig. 1(c) is $d \cong 0.88$ (for $10^{-12} \leq \epsilon \leq 0.1$), while $d \cong 0.95$ (for $10^{-12} \leq \epsilon \leq 10^{-8}$) is obtained for the range in Fig. 1(d).

We now present results for numerically determined values of the uncertainty dimension for (2) as a function of λ. In Fig. 3, we show two types of data. The dots are obtained with ranges $\Delta x \geq 10^{-7}$ and the triangles with $\Delta x \leq 10^{-9}$. In the hyperbolic case (λ larger than about 6.5), both types of data yield comparable results for d. The computed dimension values for the hyperbolic range of λ can be significantly lower than the values in the nonhyperbolic range, where the computed dimension values for the triangles are typically larger than 0.95. As expected from our previous discussion, and in contrast with the results for the hyperbolic range of λ, the triangles, which correspond to narrower ranges in x_0, are always significantly above the dots when λ is in the nonhyperbolic range. This is consistent with the supposition that, throughout the nonhyperbolic range, the measured value of d will approach 1 as smaller scales are examined.[12]

In conclusion, by using the quadratic map as an example, evidence has been presented which strongly supports the conjecture that fractal chaotic scattering sets have dimension 1 in the nonhyperbolic case. In contrast, the fractal dimension in the hyperbolic case can be far below 1. In the hyperbolic case, the structure of the time-delay function is that of a standard ($d < 1$) Cantor set, as discussed, for example, in Refs. 2 and 3(b); while in the nonhyperbolic case, it resembles the time-delay function obtained from our $d = 1$ Cantor-set model.

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[7]We emphasize that there are many other model Cantor-set constructions that one could envision. Furthermore, our models are meant to give only the gross character and not details of the behavior. In particular, Cantor sets for chaotic scattering are not, in general, self-similar.

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[11]Note that the dimension of a subset cannot exceed the dimension of the set. Thus, if the calculated dimension over a finite range of $\ln\epsilon$ approaches 1 as we zoom in further and further, the dimension in the larger set must also be 1.

[12]We have also obtained additional data on a finer grid in λ than shown in Fig. 3. These data reveal that there are pronounced narrow dips in the estimated d values for the range of scales for the dots at $\lambda \cong 4.0$ and 5.1. The $\lambda \cong 4.0$ dip corresponds to the disappearance of the period-four island that surrounds the central KAM region of the stable fixed point. Similarly the dip at $\lambda \cong 5.1$ occurs at the period-three bifurcation that destroys the central island. Zooming in to smaller scales we have verified that our estimate of d at these dips increases.