Onset of Defect-Mediated Turbulence

J.-P. Eckmann[(a)] and I. Procaccia
Department of Chemical Physics, The Weizmann Institute of Science, 76100 Rehovot, Israel
(Received 21 May 1990)

Combining methods and ideas of dynamical systems theory with the usual stability analysis for extended hydrodynamic systems we show that defect-mediated turbulence is a generic consequence of a set of physical properties which are shared by many systems. We show how the interplay between broken continuous symmetries and the dynamics of patterns leads to a universal scenario for the onset of this type of turbulence.

PACS numbers: 47.25.−c

The ubiquity of defect-mediated turbulence in hydrodynamic systems with large aspect ratios indicates that the phenomenon calls for an essentially model-free interpretation. Spatiotemporal chaos with nontrivial dynamics of defects has been observed in fluid thermal convection,[1,2] in nematics under electrohydrodynamic convection,[3-5] in surface waves,[6] in numerical simulations of certain partial differential equations in $2+1$ space-time dimensions,[7,8] etc. In this Letter we report on a theory that shows that, indeed, the phenomenon is expected to appear when rather general conditions are met. These are the following.

(i) The system exhibits a fundamental instability towards a one-dimensional cellular pattern, e.g., convection rolls. These patterns exist for a range of parameters and we shall assume that in this range no bounded, biperiodic solutions can exist.

(ii) The system has a large aspect ratio in one dimension (to be taken as an infinite x coordinate) and a medium extent in another dimension (the y coordinate). The third dimension (the z direction) is small, and it determines the scale of the cellular pattern. The extent of the system in the y direction will be a central parameter in our theory.

(iii) The underlying hydrodynamic equations possess continuous symmetries (like translation and rotation) which give rise to secondary long-wavelength instabilities of the cellular structure, e.g., skew varicose[9,10] (SV). We shall assume that the secondary instability involves transverse and longitudinal modes; this assumption will lead naturally to the creation of defects.

(iv) The highest-order spatial derivative in the hydrodynamic equations is of order m, and is linear in the fields. This assumption will allow a direct link to dynamical systems theory.

When these properties hold, we establish[11] the following generic scenario: Upon crossing the secondary instability, the fundamental cellular pattern destabilizes in favor of another, spatially biperiodic stationary solution.[12] The biperiodic stationary solution can be shown to exist within perturbation theory, but in truth it is destroyed in favor of spatially chaotic solutions by a mechanism of the Kolmogorov-Arnol'd-Moser (KAM) type. We show that when this happens, property (iii) leads naturally to the appearance of defects in the pattern. We conclude by identifying spatiotemporal chaos in such systems as a state in which defects are randomly distributed in space.

For concreteness and clarity we develop our considerations on the basis of a convenient example which contains all the needed generic features. This is the generalized Swift-Hohenberg[13] model treated in Ref. 10. It reads

$$\partial_t u+(\mathbf{U}\cdot\nabla)u=[a-(1+\partial_x^2+\partial_y^2)]^2u-u^3\,, \qquad (1)$$

where $U=(\partial_y\zeta,-\partial_x\zeta)$, and $\nabla^2\zeta=g[\nabla(\nabla^2 u)\times\nabla u]\cdot\hat{\mathbf{z}}$. Here, a and g are constants, and the terms involving ζ are used to model the coupling to the z component of vorticity in convection dynamics.[9] The phase diagram for this model is shown in Fig. 1. In the shaded region the stationary solutions are of the form[14]

$$u(x)=2\epsilon\cos(\omega x)+\mathcal{O}(\epsilon^3)\,, \qquad (2)$$

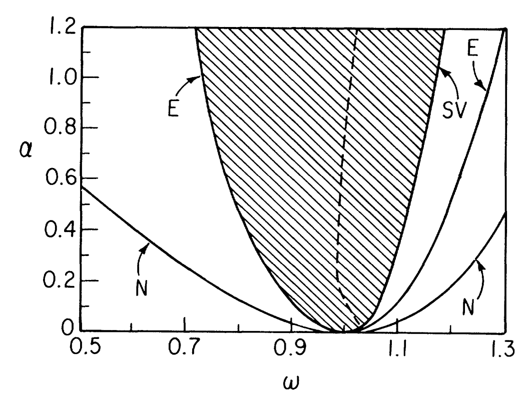

FIG. 1. The phase diagram for the model of Eq. (1), with $g=25$, from Ref. 10. The E and SV lines represent the Eckhaus and skew-varicose stability boundaries, respectively. No other long-wavelength instabilities appear between these boundaries. In the Boussinesq approximation for convection, the phase diagram is similar, except that the skew-varicose line is curved inward.

where ϵ, ω, and α are related by $\alpha=3\epsilon^2+(1-\omega^2)^2+\mathcal{O}(\epsilon^4)$.

In this notation, ω is the wave number of the one-dimensional cellular pattern required in (i). The assumption in property (iii) is realized in this model; it exhibits a skew-varicose instability. Upon crossing the SV line, the solution (2) is destabilized by long-wavelength modes of the form

$$\rho[a(\mathbf{k},\boldsymbol{\omega})e^{i\mathbf{r}\cdot(\boldsymbol{\omega}+\mathbf{k})}+b(\mathbf{k},\boldsymbol{\omega})e^{i\mathbf{r}\cdot(-\boldsymbol{\omega}+\mathbf{k})}+\text{c.c.}]\,.$$

Here, $\boldsymbol{\omega}=(\omega,0)$ and $\mathbf{k}=(k_x,k_y)$ with $k_x\neq 0$ and $k_y\ll 1$, and k_x/k_y is finite. One can show that such a mode grows exponentially in time but gets saturated by the nonlinearity. *Within perturbation theory* one can show the existence of *stationary,* spatially quasiperiodic solutions of the form

$$2\epsilon\cos(\omega x)+\rho[a(\mathbf{k},\boldsymbol{\omega})e^{i\mathbf{r}\cdot(\boldsymbol{\omega}+\mathbf{k})}+b(\mathbf{k},\boldsymbol{\omega})e^{i\mathbf{r}\cdot(-\boldsymbol{\omega}+\mathbf{k})}+\text{c.c.}]\,. \quad (3)$$

In order to see what is really happening one has to go beyond perturbation theory. This is achieved by reducing the analysis of stationary solutions to a finite-dimensional dynamical systems problem.[15] This is done rigorously using the following scheme: Consider (1) with $\partial_t u=0$. Imposing for simplicity periodic boundary conditions in the y direction, a natural basis for $u(\mathbf{r})=u(x,y)$ is

$$u(x,y)=\sum_{n=-\infty}^{\infty}u_n(x)e^{iny/h}\,, \quad (4)$$

where the width of the strip confining the system is $2\pi h$. Identify now a phase space $\mathcal{B}$ with coordinates defined by $X(x)$, where

$$X(x)=\{\partial_x^p u_n(x)\}_{n\in\mathbb{Z},\,p=0,1,2,3}\,.$$

Equation (1) is rewritten as a "dynamical" system in the form $\partial_x X=\mathbf{M}X+\mathbf{N}(X)$, with x playing the role of time. The operator $\mathbf{M}$ is infinite dimensional and is block diagonal with blocks $\mathbf{M}_n$ of the form

$$\mathbf{M}_n=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ \alpha-(1-n^2/h^2)^2 & 0 & -2(1-n^2/h^2) & 0\end{bmatrix}. \quad (5)$$

The part $\mathbf{N}(X)$ contains all the nonlinearities, and is not needed explicitly. Thus, using property (iv), we have turned the study of stationary solutions to that of an *infinite-dimensional* dynamical system. We now proceed to reduce this problem to a *finite-dimensional* one.

A straightforward calculation using the characteristic polynomial of M_n shows that bounded eigenmodes can only exist when n satisfies the inequality

$$1\pm\sqrt{\alpha}>n^2/h^2\,. \quad (6)$$

The center manifold theorem[16,17] tells us that the relevant dynamics takes place in the finite-dimensional subspace of the coordinates X_n with n satisfying (6). In this finite-dimensional phase space, the periodic spatial solutions (2) are one-dimensional limit cycles, as seen in Fig. 2(a). Since we have a band of wave vectors of periodic solutions, there is a band of limit cycles here. By Poincaré mapping, one finds a line of fixed points, cf. Fig.

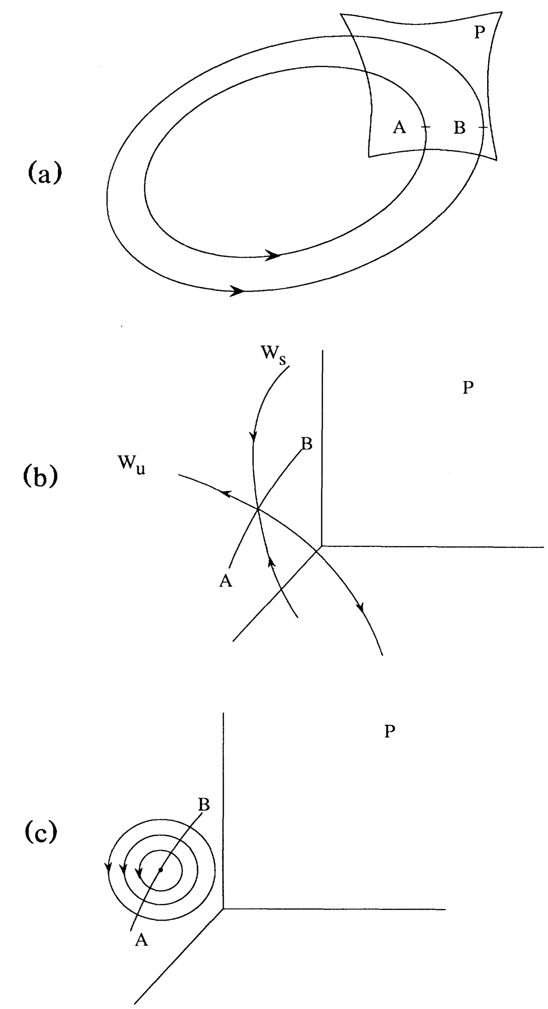

FIG. 2. The scenario for the onset of defect-mediated turbulence in its phase-space representation. (a) The phase portrait in the domain of stability towards long-wavelength perturbations. We see the periodic orbits corresponding to cellular patterns. The two limit cycles represent the boundaries of the band of allowed frequencies ω in this domain. The Poincaré section $\mathcal{P}$ has codimension one. (b) The Poincaré section $\mathcal{P}$ and the line of fixed points AB which results from the section in (a). Each fixed point has a stable manifold W_s and an unstable manifold W_u. (c) The Poincaré section $\mathcal{P}$ in the long-wavelength unstable domain. Each fixed point is now elliptic, and a few invariant circles are shown. The chaotic orbits between them are not shown.

2(b). It can be shown that each of these fixed points is hyperbolic in the parameter space where the cellular solutions are dynamically stable.[11,14,15] [The stable and unstable directions are solutions like (4), but with $\mathrm{Im}k_x \neq 0$.] On the other hand, upon crossing the secondary instability (the SV line in Fig. 1), the line of fixed point turns elliptic [Fig. 2(c); this happens when $\mathbf{k}$ becomes real]. The frequency of rotation depends nonlinearly on the amplitude [radius in Fig. 2(c)], and the "dynamical system" is effectively a nonlinear twist map. Because of the $x \to -x$ symmetry, this dynamical system is reversible, and it is known[18,19] that in reversible nonlinear twist maps the KAM mechanism sets in, leaving irrational tori intact and generating a set of chaotic solutions in between.

The conclusion is that, generically, upon crossing a secondary long-wavelength instability there exist *stationary, spatially chaotic solutions.*

At this point we want to emphasize the role of these stationary solutions in terms of the full time-dependent problem, Eq. (1). We shall argue below that these solutions have stable and unstable manifolds in function space. Accordingly, we are led to conjecture that, if these stationary states lie on the attractor, then they are important in organizing the dynamics, like hyperbolic fixed points of low-dimensional strange attractors. The time evolution will approach one of the stationary states on the stable manifold, will remain close to it for some time, and will leave it along an unstable direction.

To see the existence of a stable manifold for the time-dependent problem is easy. One examines the stability of the solution (3) (in perturbation theory) to perturbations of the amplitudes of the various components. This amounts to looking at the dynamics in the truncated equations for variations of the coefficients of the six components $e^{\pm i\boldsymbol{\omega}\cdot\mathbf{r}}$ and $e^{\pm i(\boldsymbol{\omega}\pm\boldsymbol{\kappa})\cdot\mathbf{r}}$. It is easy to see that the resulting dynamics is linearly stable, and the perturbed solution returns to the form (3). In fact, the translation invariance of the equations gives one zero eigenvalue, and the relative translation between the main frequency and the modulation gives a second zero eigenvalue. The third eigenvalue is associated with the stability of ρ about its saturation value, cf. (3). Three more eigenvalues are clearly negative.

To see that there are also unstable directions is also easy. In fact, the solution (3) is, at least close to the SV line, very close to the periodic solution (with $\rho=0$). Therefore, it has still an Eckhaus instability for (very small) $\eta > 0$ which consists of perturbations of the form $e^{\pm i\boldsymbol{\eta}\cdot\mathbf{r}}$.

The above discussion can be summarized in the following way: Cellular solutions with wavelengths belonging to the unstable domain are destabilized by long-wavelength perturbations. Close to these periodic solutions there exist other stationary solutions which are either quasiperiodic or chaotic in space. Since these stationary solutions have a stable direction for the time-dependent problem, it is reasonable to expect that the dynamics will allow a crossover from the cellular state to the quasiperiodic or spatially chaotic states. In a second step, this solution is going to evolve along its unstable direction. This has interesting consequences, as we shall see now.

During this evolution, there are, generically, changes in the total number of "rolls" (or numbers of extrema of the stationary solution). The only way that one can discard or add an extremum in the interior of the system is by going through "phase-slip" events, which in $(1+1)$-dimensional space-time amount to space-time dislocations. At such a dislocation the solution vanishes identically for a brief moment. This process of phase slip is exemplified in Fig. 3, in which we show a numerical simulation of the escape route from the solution (3) in $1+1$ dimensions. One sees that after hovering for a long time near (3), the system evolves, where every now and then a period in the solution is shed off via a process of phase slip; see also Ref. 20.

In $2+1$ dimensions, an extension of these considerations[21] shows that if it were not for the y dependence of the solutions, the solution would vanish identically on a line, $x=$const. This is topologically unstable and in fact the solutions will only vanish on spatial points, where the topological defects get born. The explanation of the precise mechanism of the nucleation of topological defects is beyond the scope of this Letter, and needs a special study. But we stress that the main conceptual feature of our approach is the realization that there is *no qualita-*

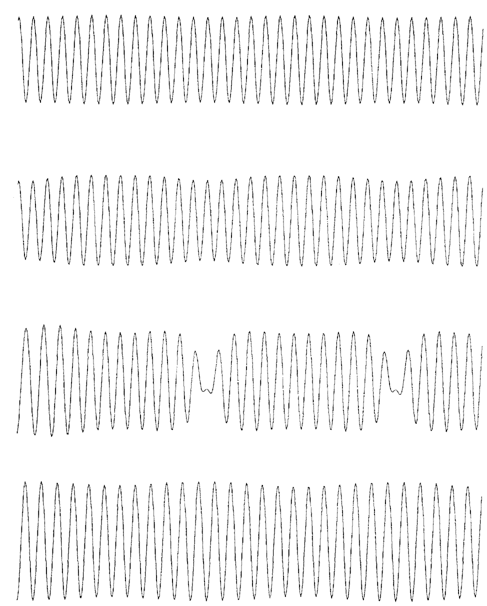

FIG. 3. Four frames at equal time-spacing for the time evolution for the Swift-Hohenberg equation, with $\epsilon=0.2$ and $\omega^2=1+1.6932\epsilon$. Note the process of phase slip.

tive difference between the appearance of space-time dislocations in the $(1+1)$-dimensional problem and spatial topological defects in the $(2+1)$-dimensional problem.

Since the states from which the evolution results in topological singularities are already spatially chaotic, the singularities will be randomly positioned in space, as is seen in defect-mediated turbulence, so that the space-time chaos is a direct consequence of the space disorder of stationary solutions.

Finally, we stress that our analysis makes no use of any specific properties of (1). Therefore, the scenario which we describe here will apply generically in systems satisfying (i)-(iv).

We would like to thank S. Alexander, G. Goren, and R. Zeitak for very pleasant and helpful discussions on the subject of this paper, and E. Braun, S. Rasenat, and V. Steinberg for discussing their experiments with us. We thank C. E. Wayne for useful information on the status of the KAM theory. J.P.E. was partially supported by the Einstein Center of Theoretical Physics at the Weizmann Institute. This work was partially supported by GIF, the German-Israeli Foundation, Grant No. I-97-169.7188, and the Fonds National Suisse.

[a]On leave from the University of Geneva, 1211 Geneva, Switzerland.

[1]G. Ahlers and R. P. Behringer, Prog. Theor. Phys. Suppl. **64**, 186 (1979).

[2]A. Pocheau, V. Croquette, and O. LeGal, Phys. Rev. Lett. **55**, 1099 (1985).

[3]R. Ribotta and A. Joets, in *Cellular Structures and Instabilities,* edited by J. E. Wesfried and S. Zaleski (Springer, Berlin, 1984).

[4]I. Rehberg, S. Rasenat, and V. Steinberg, Phys. Rev. Lett. **62**, 756 (1989).

[5]G. Goren, I. Procaccia, S. Rasenat, and V. Steinberg, Phys. Rev. Lett. **63**, 1237 (1989).

[6]J. P. Gollub and R. Ramshankar, in *New Perspectives in Turbulence,* edited by S. Orszag and L. Sirovich (Springer, Berlin, 1990).

[7]P. Coullet, L. Gil, and F. Lega, Phys. Rev. Lett. **62**, 1619 (1989).

[8]E. Bodenschatz, W. Pesch, and L. Kramer, Physica (Amsterdam) **32D**, 135 (1988).

[9]A. Zippelius and E. D. Siggia, Phys. Fluids **26**, 2905 (1983).

[10]H. S. Greenside and M. C. Cross, Phys. Rev. A **31**, 2492 (1985).

[11]J.-P. Eckmann and I. Procaccia; Weizmann Institute report, 1990 (to be published).

[12]M. Lowe and J. P. Gollub, Phys. Rev. Lett. **55**, 2575 (1985).

[13]J. Swift and P. C. Hohenberg, Phys. Rev. A **15**, 319 (1977).

[14]P. Collet and J.-P. Eckmann, *Instabilities and Fronts in Extended Systems* (Princeton Univ. Press, Princeton, NJ, 1990.

[15]Y. Pomeau and S. Zaleski, J. Phys. (Paris) **42**, 515 (1981).

[16]K. Kirchgässner, Adv. Appl. Mech. **26**, 135 (1988).

[17]J.-P. Eckmann and C. E. Wayne, Commun. Math. Phys. (to be published).

[18]J. Moser, *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Mechanics* (Princeton Univ. Press, Princeton, NJ, 1973).

[19]M. B. Sevryuk, *Reversible Systems,* Lecture Notes in Mathematics Vol. 1211 (Springer, Berlin, 1986).

[20]L. Kramer and W. Zimmerman, Physica (Amsterdam) **16D**, 221 (1985).

[21]G. Goren, J.-P. Eckmann, and I. Procaccia, Weizmann Institute report, 1991 (to be published).