PHYSICAL REVIEW LETTERS

VOLUME 66 | 18 FEBRUARY 1991 | NUMBER 7

Geometric Phase Shifts under Adiabatic Parameter Changes in Classical Dissipative Systems

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(Received 26 October 1990)

It is shown that a phenomenon analogous to Berry's phase and Hannay's angle occurs in dissipative systems. Adiabatic transport of a dissipative oscillatory system about a closed path in parameter space produces a geometric shift in the variable parametrizing the limit cycle. This quantity is written as the integral of a two-form over a surface bounded by the parameter-space loop.

PACS numbers: 03.20.+i

There has been a great deal of interest recently in the phenomena of geometric phase shifts,[1] as exemplified by Berry's phase[2] and the Hannay angle.[3] The starting point for either is a dynamical system whose Hamiltonian depends on a number of parameters. When these parameters are allowed to vary adiabatically and return to their original values, tracing out a closed path ∂M in the parameter space, a shift in the value of the appropriate cyclic dynamical variable (beyond the dynamic phase shift given by the integral of the "instantaneous" frequency) will have occurred. The phase shift so produced is a geometric quantity, by which is meant that its value is given by the integral of a two-form over any surface M bounded by ∂M. For the Berry phase, a quantum-mechanical phenomenon, the relevant variable is the phase of the wave function. For the Hannay angle, which is classical, it is the angle variable conjugate to the action. For both of these, the result is dependent upon the existence of an adiabatic invariant in a related variable: the quantum number(s) and the action, respectively.

We have found that dissipative oscillatory systems, which have neither Hamiltonians nor adiabatic invariances, nevertheless display analogous geometric phase shifts. Such systems have attracting limit cycles to which any point sufficiently near will relax. The phase variable here is the equal-time parametrization of this limit cycle. Our strategy will be to formulate the problem for dynamics on the circle, and then to embed this system in a higher-dimensional space; the circle becomes a limit cycle and the dynamics is extended into the orthogonal manifold.

An adiabatically varying first-order system on the circle is given by

$$d\theta/dt = \Omega(\theta,\mu(\epsilon t)) , \tag{1}$$

where Ω is bounded away from zero and is 2π periodic in θ, and $\mu(\epsilon t)$ is a slowly varying vector of parameters. The adiabatic limit corresponds to $\epsilon \rightarrow 0$. We want an equal-time parametrization of the circle, which we may get by writing

$$\varphi = \Phi(\theta,\mu) \equiv \omega(\mu) \int_0^\theta d\psi\, \Omega(\psi,\mu)^{-1} , \tag{2}$$

where $\omega(\mu)$ is the inverse period were μ held fixed. Thus Eq. (1) becomes

$$d\varphi/dt = \omega(\mu) + \epsilon\dot{\mu}\cdot\nabla_\mu \Phi(\theta,\mu) , \tag{3}$$

with $\epsilon\dot{\mu} \equiv (d/dt)\mu(\epsilon t)$. The first term on the right-hand side in Eq. (3) gives rise to the dynamic phase. The latter term gives rise to the phase shift, defined by

$$\Delta\varphi = \epsilon \int_0^\tau dt\, \dot{\mu}(\epsilon t)\cdot\nabla_\mu \Phi(\theta(t),\mu(\epsilon t)) ,$$

where τ is the time to complete the parameter-space circuit. We may evaluate this expression up to $O(\epsilon)$ by angle averaging the integrand over φ (see, e.g., Ref. 4), using Eq. (2). Defining the one-form

$$\chi(\mu) \equiv d\mu \cdot \left\{ \frac{\omega(\mu)}{2\pi} \int_0^{2\pi} \frac{d\theta}{\Omega(\theta,\mu)} \nabla_\mu \Phi(\theta,\mu) \right\},$$

we have, utilizing Stokes's theorem, and the closedness of the parameter-space path,

$$\Delta\varphi = \int_M d\chi(\mu) + O(\epsilon). \tag{4}$$

This is the geometric form promised.

We are now prepared to consider higher-dimensional systems. For the sake of clarity, we will work explicitly in two dimensions, though with care, higher-dimensional systems may be handled similarly.[5] For expository convenience we will assume that the intersection of the interiors of the family of limit cycles generated as μ varies over M is nonempty, so that we may adopt polar coordinates and place the origin in this common interior. Now for each μ, the limit cycle is given by $r = R(\theta;\mu)$. We are interested in fluctuations about the limit cycle, so we change to the variable $x = r - R(\theta;\mu)$ and the differential equations describing the dynamics will be of the form

$$\begin{aligned} dx/dt &= -\Lambda f(x,\theta;\mu) - \epsilon\dot{\mu}\cdot\nabla_\mu R(\theta;\mu), \\ d\theta/dt &= \Omega(x,\theta;\mu), \end{aligned} \tag{5}$$

where $f(0,\theta;\mu) = 0$, and we have introduced the parameter Λ which effectively scales the relaxation time for return to the attractor. The limiting case $\Lambda \to \infty$ clearly reduces back to the one-dimensional case, so we are moved to expand x in $1/\Lambda$:

$$x(t,\Lambda) = \sum_{m=1}^{\infty} x_m(t)\Lambda^{-m} + \xi(t,\Lambda).$$

A basic assumption is that the last term, which contains information about the initial conditions, decays like $\exp(-a\Lambda t)$ for some $a > 0$; since we will be considering indefinitely long times, this term will be neglected. We will evaluate the coefficients x_m by substituting this expansion into the first of Eqs. (5), equating coefficients, and discarding terms of second and higher order in ϵ, which cannot contribute in the adiabatic limit. There are two crucial observations. First, terms arising with factors $\partial^n f/\partial x^n$ will be multiplied by x^n. Since x is $O(\epsilon)$

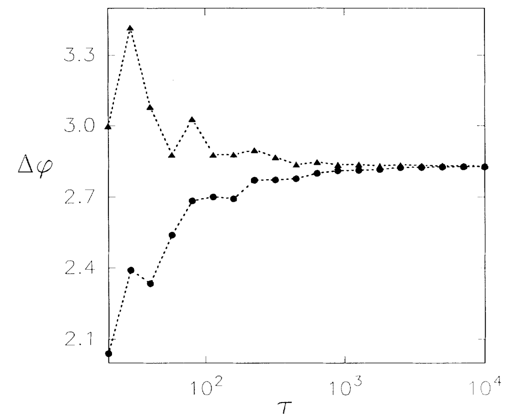

FIG. 1. $\Delta\varphi$ for $\psi = 2\pi t/\tau$ (circles), and $-\Delta\varphi$ for $\psi = -2\pi t/\tau$ (triangles), vs τ, the time to complete the circuit, in units of the oscillator period. Both figures refer to the system of equations (6) for which we have taken $\rho = 0.7979$.

these may be neglected. This simply means that we never get very far from the limit cycle. Second, note that

$$dx_m/dt = (\epsilon\dot{\mu}\cdot\nabla_\mu + \dot{\theta}\,\partial/\partial\theta)x_m.$$

The first operator on the right-hand side produces only terms which vanish in the adiabatic limit, so we keep only those generated by the second. Together, these two considerations prevent any expressions like $\ddot{\mu}$ or $\dot{\mu}^2$ from occurring in the expression for x, and these are precisely the terms that would prevent our writing $\Delta\varphi$ in the form of Eq. (4). We have

$$x_1 = -\epsilon(\partial f/\partial x)^{-1}\dot{\mu}\cdot\nabla_\mu R(\theta;\mu),$$

and utilizing the second of Eqs. (5),

$$\Omega(0,\theta;\mu)\,\partial x_m/\partial\theta = x_{m+1}\,\partial f/\partial x(0,\theta;\mu),$$

for $m \geq 2$. Since each term in the series will be of the form $\epsilon\dot{\mu}$ times some function of θ and μ, we may pull out the $\epsilon\dot{\mu}$ and sum the remaining part, writing

$$x = \epsilon\dot{\mu}\cdot\zeta(\theta,\mu,\Lambda).$$

Now we may substitute this expression into the latter of Eqs. (5) and our one-form becomes

$$\chi(\mu) = d\mu \cdot \left\{ \frac{\omega(\mu)}{2\pi} \int_0^{2\pi} \frac{d\theta}{\Omega(\theta;\mu)} \left[\nabla_\mu \Phi(\theta;\mu) + \omega(\mu) \frac{\partial \ln \Omega(\theta;\mu)}{\partial x} \zeta(\theta,\mu,\Lambda) \right] \right\},$$

where all expressions are to be evaluated at $x = 0$. Our expression for this one-form contains the original one-dimensional expression and a series of new terms arising from fluctuations about the limit cycle, and again $\Delta\varphi$ is given by Eq. (4).

To illustrate these findings we have numerically integrated the following system of equations:

$$dr/dt = r\,\partial R/\partial\theta + (R - r), \quad d\theta/dt = r, \tag{6}$$

where $R = 1/[1 + \rho\cos(\theta + \psi)]$. This system has the interesting feature that the period is independent of the parameters

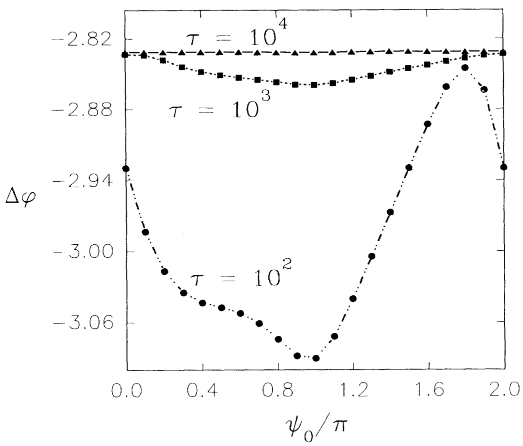

FIG. 2. $\Delta\varphi$ for $\psi=\psi_0+2\pi t/\tau$ vs ψ_0 at three different values of τ.

ρ and ψ, so its dynamic phase is just t, yet it develops a phase shift nevertheless. Clearly, for a phase shift of the form of Eq. (4), the value obtained by completing a circuit in one direction must be opposite to that found in going around the other way. Figure 1 shows $\Delta\varphi$ obtained letting ψ vary from 0 to 2π compared to $-\Delta\varphi$ going from 0 to -2π, in a time τ cycles long. For τ relatively small, $\Delta\varphi$ fluctuates strongly, and the above-mentioned relationship does not hold. As τ grows longer, both phase shifts settle to the same value, as expected. Further, if Eq. (4) is valid, we expect that it will make no difference what point on the parameter-space circuit is chosen as initial point. Figure 2 shows that for this system, it does in general depend on the initial point, but that as τ grows this dependence vanishes, as predicted. Further numerical experiments and explorations in more complex systems will be published elsewhere.[6]

In summary, we have shown that the phenomenon of geometric phase shifts under adiabatic transport around a closed curve in parameter space is more ubiquitous than previously realized; that in addition to the Hamiltonian systems studied up to now, it occurs in systems that have neither a Hamiltonian, nor any adiabatic invariances, but which do have dissipative relaxation to a limit cycle. Given the form of the argument presented here we might be led to conjecture that in any system where the notion of phase shift after transport about a closed loop in parameter space is meaningful, such a phase shift will be geometric.

We would like to thank Hugh Pendleton, Larry Abbott, and Irv Epstein for valuable discussions and suggestions. This work was supported by the National Institutes of Health under Grant No. T32NS07292 (T.B.K.) and the National Institutes of Mental Health under Grant No. MH46742 to Eve Marder (Brandeis University).

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