Average Persistent Current in a Mesoscopic Ring

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We present a microscopic calculation of the disorder-averaged persistent current for a mesoscopic ring with a fixed number of electrons. Using an idea by Imry, we show that the even Fourier components of the average current dominate and are related to the typical single-level current $\langle I_{2p}\rangle_D \propto \langle i_p^2\rangle_D/I_0$, which can be calculated by a Green's-function-averaging technique. For the small even harmonics p, the result simplifies to $\langle I_{2p}\rangle_D \approx I_0/2\pi M$, where M is the number of transverse channels. Its relevance to multiring experiments is discussed.

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Quantum persistent currents[1] in small nonsuperconducting rings threaded by a magnetic flux φ have aroused excitement and controversy during the past year.[2-8] In particular, the disorder average of the current and its harmonic content, $\langle I\rangle_D = \sum_p \langle I_p\rangle_D \sin(2\pi p\varphi)$, are intriguing theorists.[6-9] One reason is the averaging controversy concerning the differences between averaging at a constant number of electrons N versus constant chemical potential μ. The other reason is that the first persistent-current experiment was performed on an ensemble of 10^7 disconnected rings,[4] so that a comparison with theory requires one to compute the ensemble-averaged current per ring, which is different in magnitude *and* periodicity as function of flux from that of the typical current[2] in a single-ring experiment. The average current contains single-electron and collective contributions, the latter[5] due to the electron-electron interaction. Their relative size depends on the number of effective channels M_{eff}.[7] Although the form of the single-electron contribution had been conjectured from computer simulations[6] and by a correspondence argument,[7] it resisted analytical calculation because of the averaging problems involved. Here we present a Green's-function approach to this quantity and determine its form and amplitude in the diffusive regime. The results differ from the numerical ones.[6] Our main formulas are given in Eqs. (3), (4), and (8).

Three observations led gradually to a deeper understanding of the problem and the present calculation.

(i) Calculations for one-channel loops[10] and numerical studies for multichannel rings[3] indicated that while computing fluctuations $\langle I^2\rangle_D$ is insensitive to averaging at constant N versus constant μ,[2] the average currents are different, $\langle I(N)\rangle_D \neq \langle I(\mu)\rangle_D$. Specifically, the numerical studies employing averaging over N indicated period halving also for ensembles of multichannel rings, i.e., $\langle I_{p=1}(N)\rangle_{N,D} = 0$ while $\langle I_{p=2}(N)\rangle_{N,D} > 0$.[3] This can be understood from the structure of the energy spectrum of rings.

(ii) Bouchiat and Montambaux[3] found that in the ballistic regime $\langle I_{p=2}\rangle_{N,D} = I_0$ is independent of the number of channels. Based on extensive numerical calculations[6] and a correspondence argument[7] (relating the ballistic and diffusive regimes) it was conjectured that in the diffusive regime the amplitude is related to the typical single-level current[2] and $\langle I_{p=2}\rangle_{N,D} = C(l_{\text{el}}/ML)^{1/2} I_0$, which showed that both averages, $\langle I^2\rangle_D$ and $\langle I\rangle_D$, may not depend sensitively on filling.

(iii) Imry[8] observed that expanding the thermodynamic potential Ω in terms of $\delta\mu = \mu(\varphi) - \epsilon_F$ by imposing the constant-N condition allows one to isolate a term $\Delta\Omega = \frac{1}{2}\rho(\epsilon_F)(\delta\mu)^2$ that can account for period halving. A corresponding formula is derived below without expansion.

The system is a thin-walled normal-metal ring (e.g., Cu or Au) threaded by a magnetic flux $\varphi = \phi/\phi_0$, with $\phi_0 = hc/e$. The flux is treated as an external parameter. The quantum persistent current is an equilibrium property and defined in terms of the thermodynamic potential Ω of the system,[11]

$$I = -\frac{e}{h}\frac{\partial\Omega}{\partial\varphi}. \tag{1}$$

A useful unit for the current is $I_0 = ev_F/L$. We denote the number of electrons in the ring by N; for constant N the chemical potential is flux dependent, $\mu = \mu(\varphi)$. Before averaging, the current I of a single ring is strongly sample specific, changing sign as a function of filling, with an average period M_{eff},[2] and randomly with disorder configurations.

We use the following notation. At zero temperature, the characteristic length scales are the ring circumference L, elastic mean free path l_{el}, and localization length $\xi \propto M l_{\text{el}}$. In the diffusive regime one has $l_{\text{el}} < L < \xi$. We assume that the electronic phase coherence length exceeds L. A ring with a cross-sectional area A has $M = Ak_F/4\pi$ transverse channels. There are two relevant energy scales,[2] the level spacing or inverse density of states $\Delta_M = 1/\rho(\epsilon_F)$, and the correlation or Thouless energy $E_c = \pi^2\hbar D/L^2 \propto \hbar/\tau_D$, where $\tau_D = L^2/D$ is the time for diffusion around the ring ($D = v_F l_{\text{el}}/d$). The latter can be expressed as $E_c \propto \Delta_M M_{\text{eff}}$ in terms of the effective number of channels $M_{\text{eff}} = M l_{\text{el}}/L$.

We now present the details of our calculation. It proceeds in two steps, the first leading to Eq. (4) and the second being the microscopic calculation of $\langle i_p^2\rangle_D$ resulting in Eqs. (6) and (7). The electron-electron interaction is not included.

At zero temperature and constant filling N the persistent current is given by $I(N)=-(e/h)(\partial E/\partial\varphi)_N$, in terms of the total energy of the system $E=\int_0^\mu d\omega\,\omega\rho(\omega)$, with the side condition $N=\int_0^\mu d\omega\rho(\omega)=\text{const}$. Since the chemical potential μ and the density of states $\rho(\omega)$ are periodic functions of the flux φ through the ring,

$$\mu=\sum_{p=0}^{\infty}\mu_p\cos(2\pi p\varphi)\,,$$

$$\rho(\omega)=\sum_{p=0}^{\infty}\rho_p(\omega)\cos(2\pi p\varphi)\,.$$

While $\rho_{p=0}(\omega)$ is a slowly varying function of ω, the higher harmonics change sign with an average period E_c. The variation of $\mu(\varphi)$ with disorder is also of magnitude E_c.

We introduce a parameter $\bar{\mu}$ to isolate the large flux-independent contribution to the chemical potential μ. Later we choose $\bar{\mu}\equiv\langle\mu_{p=0}\rangle_D$. Differentiating E,

$$\partial E/\partial\varphi=\mu\rho(\mu)\partial\mu/\partial\varphi+\int_0^\mu d\omega\,\omega[\partial\rho(\omega)/\partial\varphi]\,,$$

and using the side condition $dN/d\varphi=0$, we obtain the following *exact* expression for the current:

$$I=-\frac{e}{h}\left[\frac{1}{2}\rho(\mu)\frac{\partial}{\partial\varphi}(\mu-\bar{\mu})^2+\int_{\bar{\mu}}^{\mu}d\omega(\omega-\bar{\mu})\frac{\partial}{\partial\varphi}\rho(\omega)\right.$$
$$\left.+\int_0^{\bar{\mu}}d\omega(\omega-\bar{\mu})\frac{\partial}{\partial\varphi}\rho(\omega)\right]\,,\tag{2}$$

where $\mu=\mu(\varphi)$ and $\bar{\mu}$ is constant. Note that $\partial I/\partial\bar{\mu}=0$ is equivalent to $dN/d\varphi=0$. We perform the disorder average on Eq. (2). The last term is $\langle I(\bar{\mu})\rangle_D$, which has been shown to be exponentially small.[2] The second term contains nonzero correlations but contributes only to higher order, $E_c/\bar{\mu}$. The average of the first term can be factorized up to corrections of the same order. The proof uses the periodicity of $\rho_{p\geq1}(\omega)$ and factorization of disorder averages. It is subtle since there are many terms of comparable order. First, differentiate $dN/d\varphi=0$ with respect to μ, which yields the condition

$$\langle\rho_{p\geq1}(\bar{\mu})\mu_{p\geq1}\rangle_D\propto\langle\rho'_{p=0}(\bar{\mu})\rangle_D\langle\mu^2_{p\geq1}\rangle_D\,.$$

Then, note that the second term in (2) contributes terms $\langle[\partial\rho^{(n)}/\partial\varphi](\mu-\bar{\mu})^{n+2}\rangle_D$, where $n\geq1$, which have the harmonic amplitudes $\langle\rho_{p\geq1}(\bar{\mu})\mu_{p\geq1}\rangle_D E_c$. Applying the above condition then implies that their size relative to the first term is of order $\langle\rho'_{p=0}(\bar{\mu})\rangle_D E_c/\langle\rho_{p=0}(\bar{\mu})\rangle_D\propto E_c/\bar{\mu}$. Hence, the disorder-averaged current is

$$\langle I(N)\rangle_D=-\frac{e}{2h}\langle\rho(\bar{\mu})\rangle_D\left\langle\frac{\partial}{\partial\varphi}(\mu-\bar{\mu})^2\right\rangle_D$$
$$\times\left[1+O\left(\frac{E_c}{\bar{\mu}}\right)\right].\tag{3}$$

The derivation is easily generalized to finite temperature and leads then to (3) with a $\mu(\varphi,T)$. One can express the leading contribution to the average current also as

$$\langle I(N)\rangle_D=\langle\rho(\bar{\mu})\rangle_D\langle(\partial\delta F/\partial N)(\partial I/\partial N)\rangle_D\,,$$

in terms of the changes of the free energy and the total current with N at constant flux ($\delta F=F-\langle F\rangle_D$).

Next, we use Eq. (3) to calculate the single-electron contribution to the average current $\langle I(N)\rangle_D$. We replace the chemical potential and its flux derivative by a single-level energy E_n and current $i_n=-(e/h)\partial E_n/\partial\varphi$, say, of a level within E_c of the Fermi surface. This is a reasonable assumption because of level repulsion and the strong (anti)correlations in the spectrum of closed rings. Relating the Fourier amplitudes of the level energies to those of the level current, $i=\sum_{p=1}^{\infty}i_p\sin(2\pi p\varphi)$, and assuming the absence of correlations between different harmonics (which is confirmed by the microscopic calculation), one concludes

$$\langle I_{2p}\rangle_D=(M/\pi p I_0)\langle i_p^2\rangle_D\tag{4}$$

for the even harmonics.[12] The odd harmonics have exponentially small amplitudes. This remarkable relationship was first conjectured for $p=1$ in Montambaux *et al.*[6] from numerical work (however, see discussion of numerical results below) and derived in Ref. 3 for the strongly localized regime.

The square of the single-level current is insensitive to averaging at constant N or μ, and we use the Green's-function approach[2] for its calculation. Starting from the standard formula

$$I=I_0\int\frac{dE}{2\pi i}\sum_{\mathbf{k}}\frac{k_x}{k_f}[G^+(k,E)-G^-(k,E)]\,,$$

we obtain, after squaring and disorder averaging, to leading order

$$\langle i^2\rangle_D=I_0^2 2\,\mathrm{Re}\int_{\bar{\mu}-\Delta_M}^{\bar{\mu}}\frac{dE}{2\pi}\int_{\bar{\mu}-\Delta_M}^{\bar{\mu}}\frac{dE'}{2\pi}\sum_{\mathbf{k},\mathbf{k}'}\frac{k_xk'_x}{k_F^2}\langle G^+(\mathbf{k},E)\rangle_D^2\langle G^-(\mathbf{k}',E')\rangle_D^2$$
$$\times\left[D(\mathbf{k}-\mathbf{k}',E-E')+\sum_{\mathbf{p}}\langle G^+(\mathbf{k}-\mathbf{p},E)\rangle_D\langle G^-(\mathbf{k}'+\mathbf{p},E')\rangle_D D^2(\mathbf{k}-\mathbf{k}'-\mathbf{p},E-E')\right.$$
$$\left.+K(\mathbf{k}+\mathbf{k}',E-E')+\sum_{\mathbf{p}}\langle G^+(\mathbf{k}-\mathbf{p},E)\rangle_D\langle G^-(\mathbf{k}'-\mathbf{p},E')\rangle_D K^2(\mathbf{k}+\mathbf{k}'-\mathbf{p},E-E')\right].\tag{5}$$

Here D and K denote the diffusion and Cooperon contributions. Equation (5) with the limits of integration extending from $-\infty$ to $\bar{\mu}$ can be used to calculate the typical total current $\langle I^2\rangle_D^{1/2}$. The second-order terms in D and K were not considered in Ref. 2.[13]

To obtain the typical single-level current, we integrate over an energy interval that contains on average one energy level near the Fermi surface. While this procedure would be unjustified for a particular impurity configuration, it is valid after impurity averaging due to the equivalence of averaging across the across the spectrum and over all impurity configurations. The summation over the flux-dependent longitudinal components of the momentum vectors is best performed by using the Poisson summation formula. After shifting the momentum variables in the second-order term, the sum over $\mathbf{p}$ involves only Green's functions and its flux-independent part gives the dominant contribution. For rings of circumference much larger than the transverse dimensions $(L \gg L_\perp)$ only the $k_\perp = k'_\perp$ terms contribute to leading order. Performing these summations and energy integrals yields

$$\langle i^2\rangle_D = I_0^2\left(\frac{4}{\pi d}\frac{l_{\mathrm{el}}}{L}\right)^2 \sum_{p=1}^{\infty}\frac{1}{p^3}\sin^2(2\pi p\varphi)\left(3-u\frac{\partial}{\partial u}\right)\{1-e^{-u}[(1+u)\cos u+u\sin u]\}\,, \tag{6}$$

with $u=\frac{1}{2}p(\pi d/M_{\mathrm{eff}})^{1/2}$. The Fourier coefficients $\langle i_p^2\rangle_D$ in Eq. (6) exhibit different p dependence for $p \lessgtr M_{\mathrm{eff}}^{1/2}$. When $p \ll M_{\mathrm{eff}}^{1/2}$, one may expand in u and find, to fifth order in u,

$$\langle i_p^2\rangle_D = \left(\frac{I_0}{M}\right)^2\left\{\frac{1}{2}p-\frac{2}{15}p^2\left(\frac{\pi d}{M_{\mathrm{eff}}}\right)^{1/2}+\cdots\right\}. \tag{7}$$

Here the leading term is independent of disorder. Performing the sum over p in Eq. (6) yields for the typical single-level current[2] $\langle i^2\rangle_D^{1/2} = I_0(l_{\mathrm{el}}/2dML)^{1/2}$ independent of flux, when $|\varphi| \gtrsim M_{\mathrm{eff}}^{-1/2}$.

By combining these results with Eq. (4), we obtain for the even Fourier coefficients (with $p \ll M_{\mathrm{eff}}^{1/2}$) of the disorder-averaged current in the diffusive regime

$$\langle I_{2p}\rangle_D = \frac{I_0}{M}\left\{\frac{1}{2\pi}-\frac{2p}{15\pi}\left(\frac{\pi d}{M_{\mathrm{eff}}}\right)^{1/2}+\cdots\right\}. \tag{8}$$

Note that the sample-specific dependence of the current on filling N has dropped out by averaging.

The amplitude in (8) can be expressed in terms of the diffusion time τ_D around the ring, $I_0/M = d(e/\tau_D)M_{\mathrm{eff}}^{-1}$, which shows that the harmonics of the disorder-averaged current are smaller by a factor of $1/M_{\mathrm{eff}}$ than the typical current $\langle I_{p=1}^2\rangle_D^{1/2}$ for the same ring.[2] For the ratio of the harmonics of the typical single-level and total currents we obtain $\langle i_p^2\rangle_D^{1/2}/\langle I_{p=1}^2\rangle_D^{1/2} = \frac{1}{4}(p/3)^{1/2}(\pi d/M_{\mathrm{eff}})$, while the ratio of the typical currents at constant flux is[2] $\langle i^2\rangle_D^{1/2}/\langle I^2\rangle_D^{1/2} \propto (\pi d/M_{\mathrm{eff}})^{1/2}$. The result that $\langle i^2\rangle_D$ is constant as a function of flux, when $|\varphi| \gtrsim M_{\mathrm{eff}}^{-1/2}$, agrees with numerical findings[2,6] and can be explained in terms of the structure of the energy spectrum.

The discrepancy between (8) and the formula $\langle I_2\rangle_D = C(I_0/M)M_{\mathrm{eff}}^{1/2}$ (with $C \approx 0.05$) inferred from computer simulations[6] comes as a surprise and a puzzle. The u^3 term in Eq. (7) or (8) has precisely that functional dependence but its coefficient vanishes. It is not clear whether finite-size effects lead to a nonzero coefficient. One would obtain a nonvanishing coefficient if one modified the diffusion constant $D = v_F l_{\mathrm{el}}/d$, e.g., by introducing an effective dimensionality[14] $d^* \neq d$ in D but not elsewhere in the calculations, but we have not found a microscopic justification for this. However, the analytical result $\langle I_2\rangle_D/I_0 \approx 1/2\pi M$ is not inconsistent with the numerical amplitudes in Fig. 3(c) of Ref. 6 for the largest M, i.e., $M = 64{,}100$. Recent simulations involving rings with larger transverse size $L_\perp$ are better fitted by (8).[15]

The leading term in Eq. (8), $\langle I_{2p}\rangle_D \approx I_0/2\pi M$, was independently obtained by Schmid[16] and Altshuler, Gefen, and Imry[17] from a formula that relates the ensemble-averaged current to the grand-canonical fluctuation of the number of particles with flux,

$$\langle I(N)\rangle_D = -(e/2h)\langle\rho(\bar{\mu})\rangle_D^{-1}\partial\langle(\delta N)^2\rangle_D/\partial\varphi\,.$$

From this they may also determine the temperature corrections, which are similar to those for the collective contribution to the persistent current.[5] However, their calculation can only produce the first term in (8), which explains, e.g., the pathological flux dependence $\langle I\rangle_D \propto I_0\cot(\pi\varphi)$.[16] This difficulty is presumably related to the expansion of the $N=$const condition to first order in $\delta\mu(\varphi)$, $\langle\rho(\bar{\mu})\rangle_D\delta\mu(\varphi) \approx -\int_0^{\bar{\mu}} d\omega\,\delta\rho(\varphi,\omega)$,[18] which is problematic since the discreteness of the energy spectrum leads to different flux dependences on the left- and right-hand sides of this equation. By avoiding such an expansion our approach yields the full harmonic content of the average current.

Equation (8) with $d=3$ applies to multiring experiments.[4] The additional ensemble average over N is trivial since the result for the average current (in units of I_0) does not depend on filling. Our formula implies that at zero temperature the ratio of the single-electron and collective[5,7] (due to the electron-electron interaction) contributions to the persistent current is $\langle I_{p=2}^{(s)}\rangle_D/\langle I_{p=2}^{(c)}\rangle_D = (3/4\tilde{\lambda})M_{\mathrm{eff}}^{-1} \approx 7M_{\mathrm{eff}}^{-1}$ for Cu or Au rings $(\tilde{\lambda} \approx 0.1)$ which indicates that the collective contribution dominates in systems with large M_{eff}.[5,7] This estimate uses the effective coupling constant $\tilde{\lambda} = \lambda/[1+\frac{1}{2}\lambda\ln(\bar{\mu}/E_c)] \approx \lambda/5$, due to higher-order diagrams,[19] rather than λ as in Ref. 5. Taking parameters appropriate for the experiment by Lévy *et al.*[4] ($L \approx 2.2$ μm, $M \approx 17\,000$, and $l_{\mathrm{el}} \approx 0.02$ μm), one predicts for the collective contribution $\langle I^{(c)}_{p=2}\rangle_D = (4l_{\mathrm{el}}/\pi d L)\tilde{\lambda} I_0 \approx 0.05$ nA and for the single-particle contribution $\langle I^{(s)}_{p=2}\rangle_D = I_0/2\pi M \approx 1$ pA, in contrast to the experimental result of ≈ -0.4 nA. It has been mentioned[4] that spin-orbit scattering may change the sign of the single-electron contribution. The present calculation shows that this is not the case, cf. Eq. (4), only the amplitude is reduced.

In summary, we have presented a microscopic calculation of the ensemble-averaged persistent current in multichannel rings in the diffusive regime. The difficulties of performing disorder averages at a constant number of electrons N have been circumvented by first relating the average total current to the fluctuations in the single-level current and then calculating the latter quantity using Green's functions in the standard way.

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