Non-Abelian Statistics in the Fractional Quantum Hall States

X. G. Wen
School of Natural Sciences, Institute of Advanced Study, Princeton, New Jersey 08540
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The fractional quantum Hall states with non-Abelian statistics are studied. Those states are shown to be characterized by non-Abelian topological orders and are identified with some of the Jain states. The gapless edge states are found to be described by non-Abelian Kac-Moody algebras. It is argued that the topological orders and the associated properties are robust against any kind of small perturbations.

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It has become clearer and clearer that the ground states of strongly interacting electron systems may contain very rich structures[1-6] which cannot be characterized by broken symmetries and are called the topological orders.[2] Physical characterizations of the topological orders are discussed in Refs. 2 and 6. It has been shown that the fractional quantum Hall (FQH) states, the chiral spin states, and the anyon superfluid states contain nontrivial topological orders characterized by the Abelian Chern-Simons (CS) theories.[2,3,5,7] It would be interesting to know whether or not the non-Abelian topological orders characterized by the non-Abelian CS theories[8] can be realized in strongly interacting electron systems. In this paper we will construct some FQH states which contain non-Abelian topological orders. The effective theory of these states is shown to be non-Abelian CS theory and the quasiparticles carry non-Abelian statistics.[8] We will also discuss how to understand the non-Abelian statistics in terms of the electron wave function.

Different electron wave functions with filling fraction $1/2n$ have been constructed in Ref. 4. The quasiparticles in these states were shown to have non-Abelian statistics, provided that these states are incompressible and the quasiparticles have finite size for a local Hamiltonian.

The spirit of our discussion is very similar to that in the mean-field approach to the spin-liquid states.[9,10] A similar construction is also used to study the SU(N) spin chains.[11] Consider a two-dimensional spinless (i.e., spin-polarized) electron system in strong magnetic field with filling fraction $\nu=M/N$. For convenience we will put the electron system on a lattice; thus the electron Hamiltonian has the form

$$H=\sum_{ij}[t_{ij}e^{ieA_{ij}}c_i^\dagger c_j+V_{ij}n_in_j]\,, \qquad (1)$$

where A_{ij} is the electromagnetic gauge potential on the lattice and $n_i=c_i^\dagger c_i$. To construct a FQH state with a non-Abelian topological order, we would like to break each electron into N partons[12] ψ_a each carrying electric charge e/N:

$$c=\psi_1\psi_2\cdots\psi_N=\frac{1}{N!}\sum_{ab\cdots c}\epsilon_{ab\cdots c}\psi_a\psi_b\cdots\psi_c\,, \qquad (2)$$

where ψ_a are fermionic fields and N is odd. After substituting (2) into (1) and making a mean-field approximation we reach the following mean-field Hamiltonian:

$$H_{\text{mean}}=\sum_{ij}t_{ij}e^{ieA_{ij}/N}U_{ij,ab}\psi_{ia}^\dagger\psi_{jb}\,, \qquad (3)$$

where

$$U_{ij,aa'}=e^{ieA_{ij}(N-1)/N}(1/N!)^2 \times\langle\epsilon_{ab\cdots c}(\psi_b\cdots\psi_c)_i^\dagger\epsilon_{a'b'\cdots c'}(\psi_{b'}\cdots\psi_{c'})_j\rangle\,. \qquad (4)$$

The mean-field solution U_{ij} can be obtained by minimizing the average of the Hamiltonian (1) on the ground state of H_{mean} in (3) (i.e., $E=\langle\Phi_{\text{mean}}|H|\Phi_{\text{mean}}\rangle$). Let us assume that there exists a Hamiltonian H such that the mean-field solution takes the most symmetric form $U_{ij,ab}=\eta\delta_{ab}$. In this case the mean-field Hamiltonian (3) describes N kinds of free partons in magnetic field, each with a filling fraction $\nu=M$. Thus the mean-field ground-state wave function is given by $\Phi_{\text{mean}}\{z_i^a\}=\prod_{a=1}^N\chi_M(z_i^a)$, where z_i^a is the coordinate of the ath kind of parton and $\chi_M(z_i)$ is the fermion wave function of M filled Landau levels.

Notice that the mean-field theory (3) contains many unphysical degrees of freedom arising from the breaking of the electrons into partons. In order to use the mean-field theory to describe the original electron system we need to project onto the physical Hilbert space which satisfies the constraint

$$\psi_{1i}^\dagger\psi_{1i}=\cdots=\psi_{Ni}^\dagger\psi_{Ni}\,. \qquad (5)$$

In the physical Hilbert space, different kinds of partons always move together. The bound states of the partons correspond to the original electrons. The electron ground-state wave function can be obtained by doing the projection on the mean-field wave function Φ_{mean} by setting $z_i^1=z_i^2=\cdots=z_i$, where z_i is the electron coordinate. The electron wave function obtained this way is just one of the FQH wave functions proposed by Jain.[12] In the following we will call such a state the NAF (non-Abelian FQH) state.

The wave function of the NAF state, $[\chi_M(z_i)]^N$, is the exact ground state of the following local Hamiltonian.[13] The kinetic energy in the Hamiltonian is such that the first $NM-N+1$ Landau levels have zero energy and other Landau levels have finite positive energies. One such kinetic energy is given by $\prod_{l=0}^{NM-N}[K-(l-\frac{1}{2})\omega_c]$, where $K=-(1/2m)(\partial_i - ieA_i)^2$. The two-body potential in the Hamiltonian has the form $V(r) \propto \partial^{N-1}\delta(r)$. One can easily see that the Hamiltonian is positive definite, and the NAF state has zero energy because the electrons in the ground state all lie in the first $NM-N+1$ Landau levels and the ground-state wave function has Nth-order zeros as $z_i \to z_j$. However, it is not clear whether the state has the *highest* filling fraction among the zero-energy states (this is related to the incompressibility). We can only show that among the Jain states[12] the NAF state is the zero-energy state with highest filling fraction. We do not know whether it is sufficient to consider only the Jain states. It would be interesting to numerically test the incompressibility of the NAF state for the above Hamiltonian. Numerical calculations have been done only for the projection onto the first two Landau levels.[14] In this case one indeed finds the Jain $\frac{2}{5}$ state to be the exact *incompressible* ground state.

The projection, or the constraint (5), can be realized by including a gauge field. Notice that under local SU(N) transformations $\psi_{ai} \to W_{i,ab}\psi_{bi}$, $W_i \in \mathrm{SU}(N)$, the electron operator c_i in (2) is invariant. Thus the Hamiltonian contains *a local* SU(N) symmetry after we substitute (2) into (1). The local SU(N) symmetry manifests itself as a gauge symmetry in the mean-field Hamiltonian (3). Notice that (3) is invariant under the SU(N) gauge transformation W_i: $\psi_i \to W_i\psi_i$ and $U_{ij} \to W_i U_{ij} W_j^\dagger$. The gauge fluctuation in the mean-field theory can be included by replacing the mean-field value $U_{ij}=\eta$ by $U_{ij}=\eta\exp(ia_{ij})$, where a_{ij} is a $N\times N$ Hermitian matrix. a_{ij} is just the SU(N) gauge potential on the lattice. The time component of the SU(N) gauge field can be included by adding a term[10] $\psi_i^\dagger a_0(i)\psi_i$ to the mean-field Hamiltonian. The constraint (5) is equivalent to the following constraint:[10]

$$J_\mu^I(i)=0, \quad I=1,\ldots,N^2-1\,, \tag{6}$$

where J_μ^I are the SU(N) charge and the current density. The constraint (6) can be enforced in the mean-field theory by integrating out the gauge-field fluctuation a_μ.[10] After the projection, the only surviving states are those which are invariant under the local SU(N) transformations. Those states correspond to the physical electron states.

The effective theory of the NAF state described above can be obtained by first integrating out the ψ_a field:

$$\mathcal{L}_{0\,\mathrm{eff}}=\frac{M}{4\pi N}A_\mu\partial_\nu A_\lambda\epsilon^{\mu\nu\lambda}+\frac{M}{8\pi}\mathrm{Tr}\,a_\mu f_{\nu\lambda}\epsilon^{\mu\nu\lambda}\,, \tag{7}$$

which is just the level-M SU(N) CS theory.[8] $f_{\mu\nu}$ in (7) is the strength of the SU(N) gauge field. The quasiparticle excitations in the NAF state correspond to the holes in various Landau levels of the partons. Those excitations are created by the parton fields ψ_a. After including the gauge fields, the properties of the quasiparticles are described by the following effective Lagrangian:

$$\mathcal{L}_{q\,\mathrm{eff}}=\sum\psi^\dagger\left[\left(i\partial_t+\frac{e}{N}A_0+a_0\right)-\frac{1}{2m}\left(\partial_i-i\frac{e}{N}A_i-ia_i\right)^2\right]\psi\,. \tag{8}$$

Equations (7) and (8) describe the low-energy properties of the NAF state.

The non-Abelian CS theory given by (7) and (8) has been studied in detail in Ref. 8. The quasiparticles ψ_a (which are called the Wilson lines in Ref. 8) are found to have non-Abelian statistics. In the following we will summarize some special properties associated with the non-Abelian statistics and discuss their relation to the microscopic electron wave function. Let us put the NAF state on a sphere. The ground state of (7) is found to be nondegenerate on the sphere. (On genus-g Riemann surfaces the ground states are degenerate.) Now let us create m quasiparticles and m' quasiholes using the operators ψ_{a_i} and $\psi_{a_j}^\dagger$. If we have ignored the gauge field a_μ (setting $a_\mu=0$), the Hilbert space generated by $\psi_{a_i}|_{i=1}^m$ and $\psi_{a_j}^\dagger|_{j=1}^{m'}$ would be $(\mathcal{H}_R)^m\times(\mathcal{H}_{\bar R})^{m'}$ which has $N^{m+m'}$ dimensions. Here $\mathcal{H}_R$ is the fundamental representation of SU(N) and $\mathcal{H}_{\bar R}$ is the dual of $\mathcal{H}_R$. However, after we include the gauge fluctuations and do the projection $z_i^a=z_i$, only the gauge-invariant states can survive the projection and appear as the physical states of the original electron system. In particular, all the states that transform nontrivially under global SU(N) are projected away. Thus the Hilbert space $\mathcal{H}_{mm'}$ of the physical states is contained in the SU(N)-invariant subspace of $(\mathcal{H}_R)^m\times(\mathcal{H}_{\bar R})^{m'}$:[8] $\mathrm{Inv}[(\mathcal{H}_R)^m\times(\mathcal{H}_{\bar R})^{m'}]$. In the above we have only used the global SU(N) gauge symmetry. The local gauge symmetry may further reduce the dimension of the Hilbert space. Not every (global) SU(N) singlet state can survive the projection and become a physical state. Thus the dimension of $\mathcal{H}_{mm'}$ can be less than that of $\mathrm{Inv}[(\mathcal{H}_R)^m\times(\mathcal{H}_{\bar R})^{m'}]$.

When $m=1$ and $m'=0$, there is no invariant state and the dimension of $\mathcal{H}_{10}$ is zero. When $m=m'=1$ there is only one invariant state. It is shown that such a state is always physical and $\mathcal{H}_{11}$ is one dimensional. In this case moving one particle around the other induces a Berry phase $\exp[i2\pi(N+1)/N(N+M)]$. When $m=m'=2$, $\mathrm{Inv}[(\mathcal{H}_R)^2\times(\mathcal{H}_{\bar R})^2]$ is two dimensional. It turns out that $\mathcal{H}_{22}$ is two dimensional if $M>1$ and one dimensional if $M=1$.[8] As we interchange the two particles created by ψ_{a_i}, $i=1,2$, we obtain a non-Abelian Berry phase for $M>1$. The 2×2 matrix describing the non-Abelian Berry phase is found[8] to have eigenvalues $-\exp[i\pi\times(-N+1)/N(N+M)]$ and $\exp[i\pi(N+1)/N(N+M)]$. For $M=1$ the Hilbert space $\mathcal{H}_{22}$ is one dimensional and the corresponding Berry phase is $\exp(i\pi/N)$. The later result is expected because the $M=1$ NAF state is just the Laughlin state with filling fraction $1/N$. The reproduction of the well-known results of the Laughlin states is a nontrivial self-consistency check of our theory.

Some of the above results can be easily understood in terms of the microscopic electron wave function. First we notice that the mean-field state $\Phi_{\rm mean}$ is a (global) SU(N) singlet and the NAF wave function can be expressed as $\langle 0|\prod_i c(z_i)|\Phi_{\rm mean}\rangle=[\chi_M(z_i)]^N$, where $c(z_i)$ is given by (2). The quasiparticles discussed above are described by the following electron wave function: $\langle 0|\prod_i c(z_i)\prod_{I,I'}\psi_{a_I}\psi^\dagger_{b_{I'}}|\Phi_{\rm mean}\rangle$. Since $\langle 0|\prod_i c(z_i)$ is an SU(N) singlet, it is clear that only the states in $\mathrm{Inv}[(\mathcal{H}_R)^m\times(\mathcal{H}_{\bar R})^{m'}]$ can survive the projection and give rise to nonzero electron wave functions. The dimension of the physical Hilbert space may be smaller than that of the invariant space because the electron wave functions induced from different mean-field singlet states may not be orthogonal to each other. For $m=m'=1$ the electron wave function can be obtained by the projection of the mean-field state $\psi_1(Z_1)\psi_1^\dagger(Z_2)|\Phi_{\rm mean}\rangle$. The electron wave function is nonzero and is given by $\chi_M(z_i;Z_1;Z_2)[\chi_M(z_i)]^{N-1}$, where $\chi_M(z_i;Z_1;Z_2)$ has one hole and one particle at Z_1 and Z_2. Thus $\mathcal{H}_{11}$ is one dimensional. For $m=m'=2$ the two electron wave functions $\Phi_{1,2}$ can be obtained by the projection of mean-field states $\psi_1(1)\psi_1(2)\psi_1^\dagger(3)\psi_1^\dagger(4)|\Phi_{\rm mean}\rangle$ and $\psi_1(1)\psi_2(2)\times\psi_1^\dagger(3)\psi_2^\dagger(4)|\Phi_{\rm mean}\rangle$ (which contain two singlets). Notice that locally the electron wave functions are the same near each quasiparticle no matter whether the quasiparticle is created by ψ_1 or ψ_2. More precisely the physical correlation functions, like the density correlation, are the same around each quasiparticle when the quasiparticles are well separated. This is because ψ_1 can be rotated into ψ_2 by a global SU(N) transformation, while the density correlation, being a SU(N)-invariant quantity, will not be changed. The effects of the other quasiparticles can be ignored since the correlation in the NAF states is short ranged and the other particles are far away. Thus the two electron states Φ_1 and Φ_2 should have the same local correlations and hence the same energy. Such a degeneracy is a bulk property just like the degeneracy of the FQH states on a torus.

When $M=1$ each kind of parton fills only the first Landau level. The action of $\psi_1(1)\psi_1(2)\psi_1^\dagger(3)\psi_1^\dagger(4)$ on the first-Landau-level wave function corresponds to multiplication by a factor

$$A_{22}=\sum_{i_1,i_2}\delta(Z_3-z_{i_1})\delta(Z_4-z_{i_2})\prod_{i,j\neq i_1,i_2}\frac{(z_i-Z_1)(z_j-Z_2)}{(z_i-z_{i_1})(z_j-z_{i_2})}$$

and the action of $\psi_1(1)\psi_1^\dagger(3)$ corresponds to a factor

$$A_{11}=\sum_{i_1}\delta(Z_3-z_{i_1})\prod_{i\neq i_1}\frac{z_i-Z_1}{z_i-z_{i_1}}\,.$$

After the projection the two resulting electron wave functions are given by $A_{22}(\chi_M)^N$ and $(A_{11})^2(\chi_M)^N$ which describe the same state since $A_{22}\propto(A_{11})^2$. Similar derivations apply to other values of m and m', and the physical Hilbert space $\mathcal{H}_{mm'}$ can be shown to be at most one dimensional for $M=1$. This is just the result of the non-Abelian CS theory. More detailed discussions of the structures of excitations in the NAF state will appear elsewhere.

We would like to remark that although the gauge field mediates no long-range interactions between quasiparticles due to the CS term, the quasiparticles ψ_a are not really equivalent to the "free" quarks in the absence of the gauge field. This is because the quasiparticles are dressed by non-Abelian flux which carries the SU(N) charge. Thus it is conceivable that when $M=1$ the quasiparticles behave like Abelian anyons with no internal degree of freedom, as has been shown in the above discussion.

Now let us discuss another fascinating property of the NAF state—the gapless edge excitations[15,6] in the NAF state. We will follow the discussions in Ref. 6. First let us ignore the constraint (6) and set $a_\mu=0$ in the mean-field theory. In this case the edge excitations are those of the integer quantum Hall states[15] described by

$$\mathcal{L}=\sum_{\alpha a}i\lambda^{\alpha a\dagger}(\partial_0-v\partial_x)\lambda^{\alpha a}\,, \tag{9}$$

where $\lambda^{\alpha a}$ is a fermion field describing the edge excitations of the αth Landau level of the ath kind of parton. The Hilbert space of (9) can be represented[11] as a direct product of the Hilbert spaces of a U(1) Kac-Moody (KM) algebra,[16] a level-N SU(M) KM algebra, and a level-M SU(N) KM algebra. This decomposition is a generalization of the spin-charge separation in the 1D Hubbard model. Notice that the total central charge of the above three KM algebras is

$$1+\frac{N(M^2-1)}{M+N}+\frac{M(N^2-1)}{N+M}=MN$$

which is equal to the central charge of (9). The above three KM algebras are generated by currents $J_\mu=eN^{-1}\times\lambda^{\alpha a\dagger}\partial_\mu\lambda^{\alpha a}$ (the electric current), $j^I_\mu=t^I_{\alpha\beta}\lambda^{\alpha a\dagger}\partial_\mu\lambda^{\alpha\beta}$, and $J^I_\mu=T^I_{ab}\lambda^{\alpha a\dagger}\partial_\mu\lambda^{ba}$, where t^I [T^I] are the generators of the SU(M) [SU(N)] Lie algebra. The currents in the SU(N) KM algebra are just the currents in (6) which couple to the SU(N) gauge field a_μ. To obtain the physical edge excitations in the electron wave function, we need to do the projection to enforce the constraint (6). Because of the above decomposition, the projection can be easily done by removing from the Hilbert space of (9) the states associated with the SU(N) KM algebra.[6] The remaining physical edge states are generated by the

U(1)×SU(M) KM algebra. The central charge of the U(1)×SU(M) KM algebra is given by $c=M(MN+1)/(M+N)$ are the specific heat (per unit length) of the edge excitations[17] is $C=c(\pi/6)T/v$. The electron creation operator[6] on the edge is given by $c=\lambda^{1\alpha_1}\cdots\lambda^{N\alpha_N}$ which has a propagator $(x-vt)^{-N}$ along the edge. We would like to point out that in general the edge excitations may have several different velocities in contrast to what was implicitly assumed above.

The above construction can be easily generalized in a number of directions: (a) We may decompose electrons into partons with different electric charge. (b) We may choose a different mean-field ground state which breaks the SU(N) gauge symmetry. Actually (a) is a special case of (b).[18] The effective CS theory for (a) and (b) will in general contain several Abelian and non-Abelian gauge fields. In particular, the FQH states studied in Refs. 3 and 6 correspond to breaking the SU(N) gauge symmetry into $[U(1)]^{N-1}$ gauge symmetry. One interesting NAF state in case (a) is the $\nu=(1+\frac{1}{2}+\frac{1}{2})^{-1}=\frac{1}{2}$ state. Its non-Abelian statistics are described by the level-2 SU(2) CS theory. The electrons in such a state lie in the first three Landau levels.

We would like to argue that the NAF states studied in this paper are generic states and their non-Abelian topological orders are robust against small perturbations. (i) The non-Abelian structures in the NAF states come from the SU(N) gauge symmetry of the mean-field ground state. To destroy the non-Abelian topological orders (and the associated non-Abelian statistics) we must break the SU(N) gauge symmetry through the Higgs mechanism. This cannot be achieved unless we add *finite* perturbations. (ii) All excitations in the NAF have finite energy gap and the interactions between them have finite range. Therefore the NAF states do not have infrared divergences and it is self-consistent to assume the interactions between the excitations do not destabilize the NAF states. Thus we expect the properties studied in this paper are universal properties of the NAF states which are robust against any small perturbations. The NAF states are a new type of the infrared fixed points and the NA topological orders should appear as a general possibility for the ordering in the ground states of strongly interacting electron systems.

It is not clear under what conditions the NAF states might be realized in nature. However, since the NAF states are generic states, they may appear in experiments under the right conditions, especially when the electron density is low and higher Landau levels are important. The low-density FQH states are largely unexplored in experiments.

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