Relativistic Bose Gas

Jeremy Bernstein
Department of Physics, Stevens Institute of Technology, Hoboken, New Jersey 07030

Scott Dodelson
Gordon McKay Laboratory, Harvard University, Cambridge, Massachusetts 02138
(Received 1 October 1990)

We calculate the partition function for a relativistic Bose-Einstein gas. The excitation energies are identified and compared with known results in various limits. In particular, this treatment shows how the Goldstone bosons of a spontaneously broken symmetry and the quasiparticles of the nonrelativistic gas emerge as special cases of one general formula. New results are obtained for the spectrum of Goldstone modes in a medium of nonzero density, and the fully relativistic case is discussed along with a possible cosmological application.

PACS numbers: 05.30.Jp, 11.30.Qc

Several authors[1] have studied the grand partition function of a relativistic finite-temperature Bose-Einstein gas. The purpose of this Letter is to present a calculation of the grand partition function of a self-coupled "charged"[2] gas of spinless bosons—the same model studied in the previous works—that goes beyond them in its generality. All the previous results emerge as special cases of our more general expression for Z. Our starting point is the field-theoretic expression for the partition function

$$Z=\int [d\pi_1][d\pi_2][d\phi_1][d\phi_2] \times\exp\left[\int_0^\beta d\tau\int d^3x\{i\pi_a\dot\phi_a - [\mathcal{H}(\phi,\pi)-\mu Q(\phi,\pi)]\}\right]. \qquad (1)$$

The functional integral is over the real scalar fields ϕ_1 and ϕ_2 and their conjugate momenta. Since we are investigating the system at finite temperature, the integral over Euclidean time runs only from 0 to $\beta\equiv 1/k_BT$. The index a labels the two real fields, so the implicit sum in the first term is over $a=1,2$. The Hamiltonian density for the self-coupled fields

$$\mathcal{H}=\tfrac{1}{2}(\pi_a\pi_a+\nabla\phi_a\cdot\nabla\phi_a+m^2\phi^2)+(\lambda/4!)\phi^4, \qquad (2)$$

where $\phi^2\equiv\phi_1^2+\phi_2^2$. This Hamiltonian is invariant under a global O(2) [=U(1)] symmetry; associated with this symmetry is a conserved charge. For this reason, in Eq. (1) we have included a Lagrange multiplier μ—the chemical potential—times the conserved charge $Q\equiv\int d^3x\,\mathcal{Q}$. Here the charge density is

$$\mathcal{Q}\equiv\phi_1\pi_2-\phi_2\pi_1. \qquad (3)$$

The net charge in the system, Q, can be obtained by differentiating with respect to the chemical potential:

$$Q=\frac{1}{\beta}\frac{\partial}{\partial\mu}\ln Z. \qquad (4)$$

This constraint equation must be inverted to determine the chemical potential μ in terms of the given charge Q and β.

To proceed we integrate over the conjugate momenta, π_1 and π_2. Using standard techniques, we find

$$Z=N\int[d\phi_1][d\phi_2]\exp\left(-\int_0^\beta d\tau\int d^3x\{\tfrac{1}{2}[\dot\phi_1^2+\dot\phi_2^2+(\nabla\phi_1)^2+(\nabla\phi_2)^2]+V(\phi)+i\mu(\phi_2\dot\phi_1-\dot\phi_2\phi_1)\}\right), \qquad (5)$$

where N is a constant and the potential $V(\phi)$ is

$$V(\phi)=\tfrac{1}{2}(m^2-\mu^2)\phi^2+(\lambda/4!)\phi^4. \qquad (6)$$

To do this functional integral, we write $\phi_i=\phi_i^{(0)}+\phi_i^{(1)}$. The zero-order fields $\phi_i^{(0)}$ are chosen so that the argument of the exponential—the action—is minimized. Clearly, any space or time dependence of the fields leads to an increase in the action, so we can take $\phi_i^{(0)}$ constant. Thus to determine $\phi_i^{(0)}$, we need to minimize the potential as a function of ϕ^2. We find that $\phi^{(0)2}$ is nonzero only if $\mu^2>m^2$. Specifically,

$$\phi^{(0)2}=\begin{cases}0, & \text{if } m^2>\mu^2,\\ (6/\lambda)(\mu^2-m^2), & \text{if } m^2<\mu^2.\end{cases} \qquad (7)$$

To see if the U(1) symmetry is broken at a given β, we must use the constraint equation which determines μ in terms of the given charge density and temperature to see if $\mu^2>m^2$. If it is, then $\phi^{(0)}\neq 0$ and the symmetry is broken.[3]

The next step is to expand the action around $\phi_i^{(0)}$. We consider only terms quadratic in the first-order fields $\phi_i^{(1)}$, so that the functional integral becomes a simple Gaussian; this is the one-loop approximation. After some tedious algebra we find that the free energy per volume Ω is given by

$$\frac{F}{\Omega} \equiv -\frac{\ln Z}{\beta\Omega} = \text{const} + V(\phi^{(0)}) + \frac{1}{\beta}\int \frac{d^3k}{(2\pi)^3} \ln(1-e^{-\beta E_1(k)})(1-e^{-\beta E_2(k)}) + \int \frac{d^3k}{(2\pi)^3} \frac{E_1(k)+E_2(k)}{2}. \tag{8}$$

The excitation energies in Eq. (8) are

$$E_{1,2}(k)^2 \equiv k^2 + 2\mu^2 + \frac{1}{2}\left(V'' + \frac{V'}{\phi}\right)\Bigg|_{\phi=\phi^{(0)}} \pm \left[2\mu^2\left(2k^2+2\mu^2+V''(\phi^{(0)}) + \frac{V'}{\phi}\Bigg|_{\phi=\phi^{(0)}}\right) + \frac{1}{4}\left(V''(\phi^{(0)}) - \frac{V'}{\phi}\Bigg|_{\phi=\phi^{(0)}}\right)^2\right]^{1/2}. \tag{9}$$

Apart from the constant, there are three terms in our expression for the free energy. The first, $V(\phi^{(0)})$, is the "vacuum" energy density. This vanishes unless $\phi^{(0)} \neq 0$. The second term is the thermal excitation energy corresponding to the thermal energy of excitations with dispersion relations given by Eq. (9). Note that with no conserved particle number ($\mu=0$) and no symmetry breaking ($m^2>0$) the excitation energies are both $E=(k^2+m^2)^{1/2}$ as expected. The last term in the free energy is infinite; it contains the one-loop corrections to the bare mass and coupling, which must be renormalized. By taking the large-k limit of the integrand, one can show that the infinite part of this term is independent of μ, so any renormalization prescription chosen at zero density also suffices in the finite-density case.

We will now apply this formula to three different cases: (i) spontaneous symmetry breaking at finite density, (ii) the nonrelativistic Bose gas (used to model superfluid ^{4}He), and finally (iii) the fully relativistic Bose gas.

(i) Spontaneous symmetry breaking.— In this case, the fundamental Lagrangian has $m^2<0$. This means that μ^2-m^2 is always greater than 0 and therefore $\phi^{(0)}$ is always nonzero.[4] It is straightforward to show that here $(V'/\phi)|_{\phi=\phi^{(0)}}=0$ and $V''(\phi^{(0)})=2(\mu^2-m^2)$, so that the excitation energies (9) reduce to

$$E_{1,2}(k)^2 = k^2+3\mu^2-m^2 \pm [9\mu^4+\mu^2(4k^2-6m^2)+m^4]^{1/2}. \tag{10}$$

We can recover familiar results if we take the net charge to be zero so that $\mu=0$. Then taking $M^2=|m^2|$, we find $E_1(k)=(k^2+2M^2)^{1/2}$, corresponding to a particle with mass $\sqrt{2}M$. The other energy is $E_2(k)=k$. This is the Goldstone boson.

But we can also study how the dispersion relation for Goldstone bosons changes in the presence of a finite charge density. Specifically, let us consider the speed of sound of the long-wavelength modes,

$$v_s \equiv \lim_{k\to 0} \frac{\partial E_2(k)}{\partial k} = \left(\frac{\mu^2+M^2}{3\mu^2+M^2}\right)^{1/2}. \tag{11}$$

To interpret this expression— aside from noting that it reduces to the proper limit when $\mu=0$— we must express the chemical potential in terms of the given density. Although the full constraint equation, (4), determining μ is quite complicated, we need consider only the zero-order contribution from the free energy. Hence to this order,

$$Q = \frac{-\partial V(\phi^{(0)})}{\partial \mu}\Omega = \frac{6\mu}{\lambda}(\mu^2+M^2)\Omega. \tag{12}$$

Suppose for the sake of definiteness that Q is positive: more particles than antiparticles. Then this equation tells us that the chemical potential μ must also be positive. There is only one positive solution to the above cubic equation, uniquely determining μ. The exact solution is not very illuminating, but we can consider the high- and low-density limits. At high density— quantitatively $\lambda Q/6\Omega \gg M^3$— we have approximately $\mu_{\text{high}} = (\lambda Q/6\Omega)^{1/3}$ while at low density $\mu_{\text{low}} = \lambda Q/6M^2\Omega$. This information determines the speed of sound. At high density, the chemical potential is much greater than M so

$$v_s^{\text{high}} = 1/\sqrt{3}. \tag{13}$$

This is a familiar result in unfamiliar territory. A perturbation in a relativistic medium propagates with speed of sound $v_s=1/\sqrt{3}$. Our formalism tells us that, at finite temperature and high density, the Goldstone bosons propagate as if in a relativistic medium. But even if the charge density is very low, the dispersion relation is affected. Here we find

$$v_s^{\text{low}} = 1-(\lambda Q/6\Omega M^3)^2. \tag{14}$$

(ii) Nonrelativistic Bose gas.— The nonrelativistic limit can be taken when the temperatures of interest are much less than the rest mass m. For He^4, $m \simeq 4$ GeV, and the system is studied near its critical temperature $T_c \simeq 2$ K $\simeq 2\times10^{-4}$ eV, deep in the nonrelativistic regime. This means that kinetic energies ($k^2/2m$)— which are on the order of the temperature— are much smaller than the rest energy m. Further, if we think of the chemical potential as the energy required to add one particle to the system, it is clear that $\mu_{\text{NR}} \equiv \mu - m$ should also be comparable to the temperature. Therefore we can expand the general results in powers of μ_{NR}/m ($k^2/2m^2$).

This case differs from the case of spontaneous symmetry breaking discussed above because there m^2 was negative so there was only one phase (at temperatures below M/λ). Here since m^2 is positive, we must analyze the excitation energies in two different regimes, $\mu^2 < m^2$ and $\mu^2 > m^2$. First consider $\mu^2 < m^2$. In this regime $\phi^{(0)}=0$ and the excitation energies (9) reduce to

$$E_{1,2}(k)=(k^2+m^2)^{1/2}\pm\mu \rightarrow (m\pm m)+(k^2/2m)\pm\mu_{\rm NR}\,. \quad (15)$$

One of the energies $E_1(k)$ is approximately $2m$, not attainable at low temperatures. Cyrogenic laboratories cannot produce a He-anti-He pair (with energy $2m$). The second energy $E_2(k)$ is just the kinetic energy minus the nonrelativistic chemical potential, a familiar result.

Now suppose $\mu^2 > m^2$. In this case, the excitation energies are the same as those written above for spontaneous symmetry breaking [Eq. (10)] but in the nonrelativistic limit. In this limit, we find

$$E_{1,2}(k)^2=k^2+(2m^2+6\mu_{\rm NR}m+3\mu_{\rm NR}^2)[1\pm(1+k^2/2m^2-2k^2\mu_{\rm NR}/m^3-k^4/8m^4)]\,. \quad (16)$$

Again one of the energies is unattainable: $E_1(k)=2m$. But the second is

$$E_2(k)=[(k^2/2m)^2+2(k^2/2m)\mu_{\rm NR}]^{1/2}\,. \quad (17)$$

This is just the energy spectrum of quasiparticles, first discovered by Bogoliubov.[5] To make this transparent, we solve for the chemical potential. In the nonrelativistic limit, Eq. (12), which determines μ in terms of $n\equiv Q/\Omega$ and the coupling λ, becomes

$$Q/\Omega=12m^2\mu_{\rm NR}/\lambda\,. \quad (18)$$

The coupling constant is related to the scattering length a used in standard treatments of the nonrelativistic Bose gas by $\lambda=48\pi ma$, so the spectrum reduces to the familiar

$$E_2(k)=[(k^2/2m)^2+(4\pi na/m^2)k^2]^{1/2}\,. \quad (19)$$

We have discovered two very different forms for the excitation energies depending on the value of the chemical potential. When do these expressions apply? That is, when is $\mu^2>m^2$? To answer this, we must solve the full constraint equation including the one-loop corrections; i.e., Eq. (8) must be differentiated with respect to μ and the result set equal to $-n$. Note that the last term in (8), when differentiated with respect to μ, gives a finite contribution—representing quantum effects—since the infinities are necessarily independent of μ. It is instructive to perform this operation in the condensed regime, when $\mu_{\rm NR}$ is positive and $\phi^{(0)}\neq 0$. Then we find

$$n=\frac{\mu_{\rm NR}m}{4\pi a}\left[1+\frac{4a\sqrt{\mu_{\rm NR}m}}{3\pi}\right]+\int\frac{d^3k}{(2\pi)^3}\frac{k^2/2m+\mu_{\rm NR}}{E_2(k)}\frac{1}{e^{\beta E_2(k)}-1}\,. \quad (20)$$

The first term on the right-hand side represents the charge stored in the vacuum. The number density of particles in excited states is nonzero even at zero temperature due to quantum fluctuations.[6] The second term in brackets corresponds to these fluctuations and coincides exactly with the result obtained for a hard-sphere gas using the method of pseudopotentials.[7] The last term is the thermal term, representing excitations due to finite temperature. As the temperature rises, more and more charge can be accommodated in excited modes so $\mu_{\rm NR}$ decreases. At some critical temperature all the charge can be accommodated by thermal excitations, so $\phi^{(0)}$ goes to 0. We can calculate the temperature T_c at which this occurs by setting $\mu_{\rm NR}=0$ in Eq. (20); the implicit equation for T_c is then[8]

$$n=\int\frac{d^3k}{(2\pi)^3}\frac{1}{e^{\beta_c(k^2/2m)}-1}\,. \quad (21)$$

(iii) Relativistic Bose gas.—Here m^2 is positive, so without a finite density, there would be no symmetry breaking. As long as the gas has not condensed, the constraint equation, (4)—with the aid of Eqs. (8) and (9)—becomes

$$n=\int\frac{d^3k}{(2\pi)^3}\left[\frac{1}{\exp\{\beta[(k^2+m^2)^{1/2}-\mu]\}-1}-\frac{1}{\exp\{\beta[(k^2+m^2)^{1/2}+\mu]\}-1}\right]. \quad (22)$$

At very high temperatures, far above the particle's mass, the charge can be accommodated in excited modes. As the temperature drops, in order for Eq. (22) to be satisfied, μ must approach m. The critical temperature at which the symmetry is broken is determined by setting $\mu=m$ in (22). If the mass is much lower than the critical temperature, this equation can be solved explicitly to give $T_c=\sqrt{3n/m}$, valid as long as the density is much larger than m^3. For a hypothetical massless particle, the critical temperature would be infinite and the symmetry always broken since the charge always rests in the condensate.

If the temperature is beneath the T_c so that a condensate has formed, the excitation energies are again given by Eq. (10). The excitations due to quantum effects[6] can be calculated; for all reasonable values of λ, these zero-temperature effects are found not to affect the condensation, even for very large densities. The speed of sound of the Goldstone mode can be computed just as before, with $v_s^{\rm low}=(\lambda n/12m^3)^{1/2}$ and high-density speed of sound again equal to $1/\sqrt{3}$.

Finally, a cosmological oddity. Since the cosmological expansion is adiabatic, we expect any cosmological asymmetry to scale with the temperature. That is, we expect $n = etT^3$, where n is the net difference between the number of particles and antiparticles and η is a constant. At high temperatures, the density of charge that can be accommodated in thermally excited modes—the right-hand side of Eq. (22)—increases as μT^2. We have argued, though, that the left-hand side increases as T^3. This means that as the temperature increases, $|\mu|$ must also increase. Once $|\mu|$ becomes higher than m, a condensate forms and Eq. (22) no longer holds; i.e., at *high* temperatures all the charge cannot be accommodated by the excited modes and some must reside in the condensate. The symmetry is broken at high temperatures[4] but restored at low temperatures. The critical temperature at which symmetry restoration takes place is given by $T_c = m/3\eta$.

It is straightforward to extend this work to cases with more complicated dynamics. For example, finite density plays a significant role in pion condensation in neutron stars; whether or not such a phenomenon occurs can greatly influence the equation of state.[9] Finite density is also essential to the stability of nontopological solitons; investigations are underway to see if such objects remain stable even after accounting for thermal excitations. The present treatment is well suited to attack these problems among others.

This work has benefited greatly from many conversations with L. Brown. We have made use of ideas, in the nonrelativistic case, contained in an unpublished manuscript of L. Brown. Brown and S. Jeon have studied the radiative corrections to the tree approximation, one of the subjects to be taken up in a future publication. In addition, we are pleased to acknowledge the hospitality of the Aspen Center for Physics where this work was carried out. Among the people we consulted there were E. Abrahams, D. Boulware, S. Coleman, D. Fisher, J. Frieman, D. Gross, H. Haber, and E. Weinberg. We have also profited from conversations with K. Benson, G. Feinberg, P. Hohenberg, L. Widrow, and correspondence with J. I. Kapusta. S.D. would like to thank the Center for Theoretical Physics at MIT and A. Guth for their hospitality while some of this work was done. The work of S.D. is supported in part by the U.S. Department of Energy under Grant No. DE-FG02-84ER40158 with Harvard University.

---

[1]A closely related work is that of J. I. Kapusta, Phys. Rev. D **24**, 426 (1981). Kapusta's main purpose was to find the critical temperature, so he did not derive the excitation energies. Indeed, in Sec. II of that work, an approximate expression for the partition function is given, which is sufficient for the stated purpose. The excitation energies we derive can be inferred from the Appendix of Kapusta's work by finding the roots of the determinant of the inverse propagator matrix. Also of interest is the work of H. E. Haber and H. A. Weldon, Phys. Rev. D **25**, 502 (1982). [See also H. E. Haber and H. A. Weldon, Phys. Rev. Lett. **46**, 1497 (1981).] We focus on slightly different cases, however. They treat not a U(1) [=O(2)] global symmetry as we do, but a general O(N) symmetry and take N large. The nonrelativistic results, which emerge as a special case of our treatment, are well known and can be found in, for example, R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1988), Chap. 11. Another special case of these general results is a spontaneously broken symmetry at finite temperature but zero density. This was first discussed by S. Weinberg, Phys. Rev. D **9**, 3357 (1974). For a general review, see J. I. Kapusta, *Finite Temperature Field Theory* (Cambridge Univ. Press, Cambridge, 1989), especially Chap. 7.

[2]By "charge" we mean a conserved quantity, and we will use this term interchangably with "number." We are assuming that the symmetry leading to this conserved charge is global, not a gauge symmetry. Therefore, the particles interact with each other but not with any massless boson like the photon: They are not electrically charged.

[3]In the tree approximation the expectation value of ϕ is equal to $\phi^{(0)}$. Hence the symmetry is broken when $\phi^{(0)}$ is not zero. Radiative corrections, which make the expectation value of ϕ and $\phi^{(0)}$ unequal, will be examined in a future publication.

[4]This is true until the temperature gets so large ($\sim |m|/\lambda$) that the one-loop (order $\hbar$) corrections overwhelm the zero-order term.

[5]N. N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).

[6]The fully relativistic expression for the number density of excitations due to quantum effects is

$$n = \frac{\mu(\mu^2 - m^2)^{3/2}}{\sqrt{2}\, 2\pi^2 m} \times \int_0^\infty \frac{dx}{x^3} \frac{(1+x^2)/(1+2x^2)^{1/2} - 1}{\{2x^2 + [1 + 3x^2 + (1+2x^2)^{1/2}](\mu^2 - m^2)/m^2\}^{1/2}} .$$

[7]T. D. Lee, K. Huang, and C. N. Yang, Phys. Rev. **106**, 1135 (1957).

[8]A word of warning is in order here. Very close to the critical temperature the Gaussian approximation that we used to derive the partition function breaks down. Roughly, the Gaussian approximation is valid as long as fluctuations in the ϕ field are small. This is not true at (or very near) the critical temperature when the correlation length gets very large.

[9]S. L. Shapiro and S. A. Teukolsky, *Black Holes, White Dwarfs, and Neutron Stars, The Physics of Compact Objects* (Wiley, New York, 1983).