PHYSICAL REVIEW LETTERS

VOLUME 66 | 11 FEBRUARY 1991 | NUMBER 6

Dirt Roughens Real Sandpiles

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(Received 18 September 1990)

It is shown that translational disorder in the interior bulk of real sandpiles is a relevant perturbation on self-organized surface fluctuations. In two simplified models, this disorder destroys the flat phase of the surface; surface fluctuations are instead described at long wavelengths for all parameter values by a simple diffusion equation with noise, which implies a logarithmically rough surface in $d=2$. This result *suggests* that real sandpiles may also be described by a simple diffusion equation, and *proves* that bulk translational disorder is important for their surface fluctuations.

PACS numbers: 05.40.+j, 05.60.+w, 46.10.+z, 64.60.Ht

Recently, it has been suggested[1] that "sandpiles" may exhibit self-organized criticality. These sandpiles are dynamical systems consisting of aggregates of particles (e.g., grains of sand) held up in a gravitational field by the mutual contact friction between the particles. Particles are then repeatedly and randomly dropped onto the sandpile, steepening its surface. As this happens, avalanches occur with increasing rapidity, ultimately removing particles from the surface at the same average rate as they are being randomly deposited. This steady state is believed to exhibit self-organized criticality.

Subsequent theoretical analyses[2] have shown that for a large class of simplified models of this system, a phase transition occurs in $d=2$ between a "high-temperature," logarithmically rough phase characterized by diffusive ($z=2$) dynamical behavior, and a "low-temperature" smooth phase with "superdiffusive" ($z<2$) behavior.[3] Both phases display self-organized criticality with exactly calculable[2,3] exponents.

This paper argues that this phase transition, and the low-temperature smooth phase, are artifacts of the imposition, in both computer simulations[1,4] and analytic treatments,[2] of perfect translational order on the interior of the sandpile, and that both disappear in a large class of models which, like real sandpiles,[5] lack translational order in the bulk. Thus, real sandpiles will not be in the universality class of the bulk translationally ordered models previously studied.[2]

It is not clear whether real sandpiles will be in the universality class of the models studied here either, since there are some potentially important differences between the type of translational disorder in these models and that actually present in real sandpiles. These results do, however, demonstrate the importance of properly treating the effects of bulk disorder on the surface, showing that such effects are "relevant" and can radically alter the behavior of the surface, as they do for these models by destroying an entire phase.

Furthermore, if the differences between the types of disorder studied here and those in real sandpiles prove irrelevant, then the results obtained here will also apply to real sandpiles.

The differences between the models for sandpiles considered previously[2] and the two models I will consider here are illustrated in Fig. 1. Figure 1(a) illustrates the perfectly translationally ordered models treated theoretically[2] and in simulations[1,4] to date. In these models, the sandpile can be thought of as composed of a d-dimensional array of columns of identical, perfectly stacked cubes.

I treat two disordered versions of this model in this paper. In the first, weakly disordered, which is illustrated

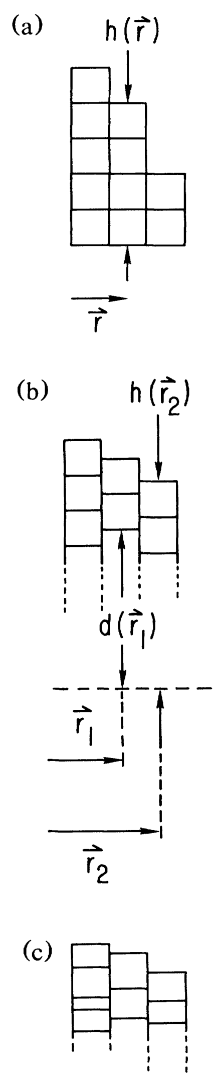

FIG. 1. (a) Illustration of the bulk-ordered models for sandpiles previously studied. Perfect cubical grains are added atop ordered columns of cubes, each in perfect registry with its neighbors. (b) The weakly disordered model studied here. Each column is still perfect, but now is displaced up or down by a quenched random, independent amount $d(\mathbf{r}) \equiv a\phi(\mathbf{r})/2\pi$. (c) The strongly disordered model. The columns themselves are no longer translationally ordered; the random vertical displacement of each column can now vary *along* the column as well.

in Fig. 1(b), the sandpile is still modeled by perfect columns of perfect cubes, but with each column independently shifted vertically by some random distance $d(\mathbf{r})$ uniformly distributed between zero and the cube length a. Here $d(\mathbf{r})$ is a *quenched* random variable, in the sense that it does not evolve in time, and is totally decorrelated with fluctuations of the interface height.

In the second, strongly disordered, model, d varies *along* the column, as well as between columns, as illustrated in Fig. 1(c). The disorder is still treated as quenched. This model is somewhat closer to the reality of true sandpiles, although, being quenched, both models still have the rather artificial feature that, if sand is removed from a region by an avalanche and then subsequently replaced by later fluctuations, the new sand sits in exactly the same (though disordered) positions as the original sand. Despite this artificiality, these models still demonstrate quite dramatically the importance of disorder.

I will begin by reviewing past work[2] on the ordered model Fig. 1(a). Clearly, in this model, the height $h(\mathbf{r})$ of the surface at any 2D lattice point $\mathbf{r}$ is an integral multiple of the cube edge length a. The system evolves[1] by adding blocks at random points on the surface (but always squarely atop a column, see Fig. 1) and then allowing blocks to hop down (along the direction of the component of gravity parallel to the surface) if the height of a column relative to those below it exceeds a critical height h_c. In the prototypal models,[1] one waited until all "avalanches" started by the addition of one random block had ceased before adding another block. The very intriguing self-organized criticality displayed by these models has been the subject of a great deal of study.[4] A totally different, but more analytically tractable class of models[2,3] continues adding blocks at a constant average rate per unit area, even during avalanches.

The important features of this latter class of models for determining the long-distance and long-time behavior are the following:[2,3] (1) the conservation law[3] that the *deterministic* part of the dynamics (the hopping) conserves the total number of grains, and hence $\int h(\mathbf{r},t)d^3r = \text{const}$ in the absence of randomly added blocks; (2) the translational symmetry[2] $h(\mathbf{r}) \to h(\mathbf{r}) + ma$, m integer, coming from the fact that all the grains are the same size, and that the dynamics only depends on height *difference*; (3) the fact that[3] the random effects (the addition of grains) violate the conservation law (1).

A continuum dynamical model which embodies these principles is

$$\partial_t h = \nu_\parallel \partial_\parallel^2 h + \nu_\perp \partial_\perp^2 h + \lambda \partial_\parallel \cos(Gh) + \eta \,, \tag{1}$$

where $\eta(r,t)$ is a Gaussian white noise with purely short-ranged correlations,

$$\langle \eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = 2D\delta^d(\mathbf{r}-\mathbf{r}')\delta(t-t') \,, \tag{2}$$

and $G = 2\pi/a$.

The cosine term reflects the $h \to h + ma$ translational symmetry, while the derivatives are required to satisfy the conservation law, and terms like $\partial_\perp \cos[\partial_\perp \cos(Gh)]$ are forbidden by $r_\perp \to -r_\perp$ symmetry. Here $\parallel$ ($\perp$) means parallel (orthogonal) to the projection of gravity $\mathbf{g}$ onto the mean plane ($\mathbf{r}$) of the surface. The properties of the dynamical model Eq. (1) are discussed elsewhere.[2,3] Its extension to include the types of disorder illustrated in Figs. 1(b) and 1(c) is

$$\partial_t h = \nu_\parallel \partial_\parallel^2 h + \nu_\perp \partial_\perp^2 h + \lambda \partial_\parallel \cos[Gh + \phi(\mathbf{r},h)] + \eta \,, \quad (3)$$

with the random noise η having the same statistics Eq. (2) and $\phi(\mathbf{r},h)$ a quenched random variable with statistics $\langle e^{i\phi(r,h)}\rangle = 0$,

$$\langle e^{i[\phi(r,h)-\phi(0,0)]}\rangle = f_\perp(r/\xi) f_z(h/\xi_z) \,. \quad (4)$$

In (4), $f_\perp(x)$ is short ranged, and falls off exponentially for $x \gg 1$. The behavior of f_z distinguishes the two models:

$$f_z(z) = \begin{cases} 1, & \text{weakly disordered model [Fig. 1(b)]}, \\ e^{-z}, & \text{strongly disordered model [Fig. 1(c)]}. \end{cases} \quad (5)$$

I begin by treating the weakly disordered model, in which ϕ is a function only of $\mathbf{r}$. My approach is to map the stochastic equation of motion (3) onto an equilibrium statistical-mechanics problem in $d+1$ dimensions.[6] The quenched disorder in this *equilibrium* problem is then treated using the replica trick, which reduces the problem to an ordered equilibrium model, which can (finally) be treated directly by standard renormalization-group techniques.

The mapping of the stochastic equation of motion (3) onto an equilibrium model using the Martin-Siggia-Rose formalism[6] is straightforward. The result is that all correlation functions of h are exactly those of a $(d+1)$-dimensional equilibrium model with the Hamiltonian

$$H(\{\pi,h,\phi\}) = \int d^d r\, dt \left(\tfrac{1}{2} D\pi^2(\mathbf{r},t) + i\pi(\mathbf{r},t)\{\partial_t h - \nu_\parallel \partial_\parallel^2 h - \nu_\perp \partial_\perp^2 h - \lambda \partial_\parallel \cos[Gh + \phi(\mathbf{r})]\}\right) + \ln J^{-1}(\{h\}) \,, \quad (6)$$

where $J(\{h\})$ is a Jacobian factor whose exact form will not be needed, $\pi(\mathbf{r},t)$ is a dummy field that is functionally integrated over in calculating the partition function, and $\phi(\mathbf{r})$ is also integrated over, but, unlike h and π, is independent of time t.

The functional integral over ϕ can now be performed perturbatively in the coupling λ using the replica trick. Using the earlier expressions for the correlations of $e^{i\phi}$, and gradient expanding, I obtain, perturbatively,

$$H_n = \sum_{\alpha=1}^{n} \int d^d r\, dt \{\tfrac{1}{2} D\pi_\alpha^2(\mathbf{r},t) + i\pi_\alpha(\mathbf{r},t)[\partial_t h_\alpha(\mathbf{r},t) - \nu_\perp \partial_\perp^2 h_\alpha(\mathbf{r},t) - \nu_\parallel \partial_\parallel^2 h_\alpha(\mathbf{r},t)]\}$$

$$+ g \sum_{\alpha,\beta=1}^{n} \int d^d r\, dt\, dt'\, \partial_\parallel \pi_\alpha(\mathbf{r},t) \partial_\parallel \pi_\beta(\mathbf{r},t') \cos\{G[h_\alpha(\mathbf{r},t) - h_\beta(\mathbf{r},t')]\} - \ln J(\{h_\alpha\}) \,, \quad (7)$$

where $g \equiv \frac{1}{4}\lambda^2 \int d^d r f_\perp(|\mathbf{r}|/\xi)$, and I have ignored all but the leading-order term in the aforementioned gradient expansion. Note that the coupling term in this expression (the term proportional to g) is infinitely long ranged in time; this reflects the infinitely long-ranged temporal correlations of the ϕ fields, which are in turn a consequence of the fact that the ϕ field is quenched and hence time independent. As always with the replica trick, properties of the quenched model are derived from the $n \to 0$ limit.

Renormalization-group recursion relations for the Hamiltonian (7) can now be constructed perturbatively in g, which is the models only relevant nonlinearity.[7] Note that there is no graphical renormalization of any of the terms bilinear in π_α and h_α, since the g term remains invariant under $\pi \to -\pi$, a symmetry which is lacked by the bilinear terms. Furthermore, the only terms quadratic in π_α generated by the g term involve two $\partial_\parallel$ derivatives as well, since the g term does. These terms are irrelevant at long wavelengths relative to the $D\pi_\alpha^2$ term already present. Thus, the parameters D, $\nu_\parallel$, and $\nu_\perp$ are renormalized only by scaling.

I will choose the renormalization-group rescalings $r_\parallel \to b r_\parallel$, $r_\perp \to b^\zeta r_\perp$, $t \to b^z t$, where the anisotropy exponent ζ and the dynamical exponent z will be chosen to produce fixed points. In addition, the fields rescale according to $\pi_\alpha \to b^{\lambda_\pi}\pi_\alpha$ and $h_\alpha \to b^{\lambda_h} h_\alpha$. Choosing to keep the coefficient of $\pi_\alpha \partial_t h_\alpha$ in H fixed at unity leads to the requirement $\lambda_h = -[\lambda_\pi + (d-1)\zeta + 1]$, which is exact since $\pi_\alpha \partial_t h_\alpha$, like all the other relevant terms in the quadratic part of H, suffers no graphical renormalization.

The renormalization-group recursion relations are

$$\frac{dD}{dl} = [(d-1)\zeta + z + 1 + 2\lambda_\pi] D \,, \quad (8)$$

$$\frac{d\nu_\parallel}{dl} = (z-2)\nu_\parallel \,, \quad (9)$$

$$\frac{d\nu_\perp}{dl} = (z-2\zeta)\nu_\perp \,, \quad (10)$$

$$\frac{dG}{dl} = -[\lambda_\pi + (d-1)\zeta + 1] G \,, \quad (11)$$

$$\frac{dg}{dl} = (z - 2 - K_d T) g + C(n-2) g^2 + O(g^3) \,, \quad (12)$$

where I have defined $T \equiv DG^2/(\nu_\parallel \nu_\perp)^{1/2}$, C is a positive constant of order unity, and K_d is the surface area of a d-dimensional sphere divided by $(2\pi)^d$. Note that the first three of these recursion relations are *exact* by virtue of the arguments just given, while that for G is also exact since G is likewise graphically unrenormalized, by translation invariance $[h(\mathbf{r},t)\to h(\mathbf{r},t)+a]$.

In order for the first three recursion relations (for $\nu_\parallel$, $\nu_\perp$, and D) to have a fixed point, we must choose $z=2$, $\zeta=1$, and $\lambda_\pi=-(1+\frac{1}{2}d)$. Using these values and taking the limit of interest $n\to 0$ in the recursion relations for G and g leads to

$$\frac{dT}{dl}=(2-d)T\,, \tag{13}$$

$$\frac{dg}{dl}=-K_d Tg-2Cg^2\,. \tag{14}$$

Since $C>0$, in all dimensions d, g renormalizes to zero in the long-wavelength ($l\to\infty$) limit, and we recover a simple isotropic diffusion equation.

Now consider the second, strongly disordered, model [Fig. 1(c)] in which the quenched phase ϕ depends on h as well as $\mathbf{r}$. This model can be mapped onto an equivalent statistical-mechanics problem and replicated in exactly the same way as the weakly disordered model. The first difference appears when the quenched averages over ϕ are evaluated perturbatively in λ. Now we have an anharmonic term in the replicated Hamiltonian of the form

$$\int_{-\infty}^{\infty}dG'\,\tilde V(G')\sum_{\alpha,\beta}\int d^dr\,dt\,dt'\,\partial_\parallel\pi_\alpha(\mathbf{r},t)\,\partial_\parallel\pi_\beta(\mathbf{r}',t)\times\cos\{G'[h_\alpha(\mathbf{r},t)-h_\beta(\mathbf{r},t')]\}\,, \tag{15}$$

where

$$\tilde V(G')\equiv g\int_{-\infty}^{\infty}\cos(G'y)\cos(Gy)f_z(y/\xi_z)\,dy$$

and g is the same as in the weakly disordered case. This model is clearly a linear superposition of (an infinite number of) terms each of which has exactly the same form as in the weakly disordered case. Therefore, all of the arguments about the exactitude of the recursion relations for D, $\nu_\parallel$, and $\nu_\perp$ go through exactly as before. In addition, the recursion relations for each $\tilde V(G')$ can be read off to linear order in $\tilde V(G')$ from the recursion relations for g and G derived in the weakly disordered case:

$$\frac{\partial\tilde V(G')}{\partial l}=-\left(1-\frac{d}{2}\right)G'\frac{\partial\tilde V(G')}{\partial G'}-\frac{K_dT}{G^2}G'^2\tilde V(G')\,, \tag{16}$$

where T is defined as before. The first term on the right-hand side of (16) arises from the changes in G' upon rescaling.

Defining $V(y)\equiv\int_{-\infty}^{\infty}(dG/2\pi)e^{iGy}\tilde V(G)$, (16) can be rewritten as

$$\frac{\partial V(y)}{\partial l}=D\frac{\partial^2V}{\partial y^2}+\frac{2-d}{2}\frac{\partial}{\partial y}yV(y)\,, \tag{17}$$

whose solution is

$$V(y,l)=e^{\varepsilon l}\int_{-\infty}^{\infty}\frac{dy'\,V(y',l=0)}{\sqrt{2\pi t(l)}}\exp\left(-\frac{yl^{\varepsilon l}-y'}{2t(l)}\right)\,, \tag{18}$$

where I have defined $t(l)\equiv D(e^{2\varepsilon l}-1)/\varepsilon$ and $\varepsilon\equiv(2-d)/2$. From (18) it is straightforward to show that for all d, and for arbitrary initial $V(y,l=0)$, $V(y,l\to\infty)$ vanishes up to an irrelevant constant for all y. Thus, the random potential term vanishes, and one is left, again, with purely diffusive behavior for the sandpile surface.

I thank Geoff Grinstein and Dung-Hai Lee for teaching me most of what I know about sandpiles, and Mehran Kardar for teaching me the rest. I am also grateful to the IBM Almaden Research Center for its hospitality while this work was being completed.

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