Octonionic Superstring Solitons

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(Received 27 November 1990)

An extended-soliton solution to the low-energy heterotic-field-theory equations of motion is constructed from an eight-dimensional octonionic instanton. The soliton describes a string in ten-dimensional Minkowski space, and preserves only one of the sixteen spacetime supersymmetries.

PACS numbers: 11.17.+y, 11.10.Kk

There have been many attempts over the years to incorporate the unique algebra of octonions into physics. However, in most cases a clear physical framework has been lacking. Superstring theory contains many hints of an underlying octonionic structure, particularly in connection with the important role of triality in SO(8). The presence of the exceptional group E_8 in the heterotic string is also extremely suggestive. Additional hints come from the presence of analogs of the Jordan algebra in the operator product algebra of fermion vertex operators,[1,2] as well as from a variety of other observations.[3]

A further connection between strings and octonions is provided by the existence of super Yang-Mills theories and Green-Schwarz superstring actions in dimension $D=3,4,6,10$, with the number of transverse dimensions (1,2,4,8) coinciding with the dimensions of the four Hurwitz algebras, R, C, H, and O. These four algebras also correspond to the four Hopf fiberings of spheres,

$$S^1 \xrightarrow[Z_2]{} S^1, \quad S^3 \xrightarrow[S^1]{} S^2, \quad S^7 \xrightarrow[S^3]{} S^4, \quad S^{15} \xrightarrow[S^7]{} S^8, \tag{1}$$

and are also intimately linked to the three parallelizable spheres, S^1, S^3, and S^7, which appear as the fibers in these maps. The first three of these correspond physically to the kink in 1+1 dimensions, the Dirac monopole in three dimensions, and the Yang-Mills instanton in four dimensions. It seems likely that the fourth Hopf map should also play an important role in physics. It is possible to use this last map to construct a Yang-Mills field configuration in eight dimensions with gauge group SO(8),[4] although the physical interpretation of this solution is unclear since it has infinite action with the standard Yang-Mills action and is not a solution of the standard Yang-Mills equation of motion. Here we will use a related eight-dimensional SO(7) Yang-Mills field configuration obeying a self-duality condition first proposed in Ref. 5 and subsequently solved in Refs. 6 and 7 which is a solution of the standard equation of motion.

In this paper we will show that this solution provides the starting point for the construction of a supersymmetric soliton string solution to the low-energy equations of motion of the heterotic string whose structure is intimately related to octonions. As in Refs. 8 and 9 we search for a solution to lowest nontrivial order in α' of the equations of motion that follow from the bosonic action

$$S=\frac{1}{2\kappa^2}\int d^{10}x\sqrt{g}\,e^{-2\phi}\left[R+4(\nabla\phi)^2-\frac{1}{3}H^2-\frac{\alpha'}{30}\mathrm{Tr}F^2\right] \tag{2}$$

which also preserves at least one supersymmetry. This means in ten dimensions that there is at least one positive-chirality Majorana-Weyl spinor ϵ satisfying the equations

$$\begin{aligned}\delta\chi&=F_{MN}\gamma^{MN}\epsilon=0\,,\\ \delta\lambda&=(\gamma^M\partial_M\phi-\tfrac{1}{6}H_{MNP}\gamma^{MNP})\epsilon=0\,,\\ \delta\psi_M&=(\partial_M-\tfrac{1}{4}\Gamma_M^{AB}\Gamma_{AB})\epsilon=0\,,\end{aligned} \tag{3}$$

where the generalized connection $\Omega_M^{AB}=\omega_M^{AB}+H_M^{AB}$. (Our notation follows that of Ref. 9 except for a change in the sign of ϕ.)

It is natural to search for a string solution invariant under SO(8) rotations in the transverse space as well as two-dimensional Poincaré transformations along the string. Singular solutions of this form preserving half of the supersymmetries were given in Ref. 8 and it was argued that these solutions correspond to the solution of the massless field equations outside a fundamental macroscopic string. Our point of view here is different in that we are searching for smooth solutions without source terms which correspond to soliton solutions to string theory. The possibility of such solutions arises as a natural consequence of the duality conjecture made in Ref. 9 relating strings and 5-branes.

The solution we will present makes use of features peculiar to eight dimensions which we now discuss. Our notation will follow that of Refs. 5 and 10 for the most part. Related results can also be found in Refs. 11 and 12. We start by picking definite commuting SO(8) spinor $\eta_\pm$ with $\Gamma^9\eta_\pm=\pm\eta_\pm$ normalized to $\eta_\pm^T\eta_\pm=1$. We can then introduce a fourth-rank antisymmetric tensor

$$c_\pm^{mnpq}=\eta_\pm^T\Gamma^{mnpq}\eta_\pm \tag{4}$$

which is self-dual or anti-self-dual depending on the chirality of η:

$$c_\pm^{mnpq}=\pm\frac{1}{4!}\epsilon^{mnpqrstu}c_{\pm rstu}\,. \tag{5}$$

There exists an explicit construction of the SO(8) gamma matrices in terms of the octonion structure constants c_{ijk} defined by

$$e_ie_j=-\delta_{ij}+c_{ijk}e_k,\quad i,j,k=1\dots7\,, \tag{6}$$

with e_i the imaginary octonions. Using this construction and an explicit choice for $\eta_\pm$ one finds

$$c_\pm^{ijk8}=c^{ijk},\quad c_\pm^{ijkl}=\pm\frac{1}{3!}\epsilon^{ijklmnp}c_{mnp}\,. \tag{7}$$

Useful identities obeyed by the tensor $c_\pm^{mnpq}$ include the following:

$$\begin{aligned}&c_\pm^{mnpq}c_{\pm mnpq}=336\,,\\&c_\pm^{mnpr}c_{\pm mnps}=42\delta_s^r\,,\\&c_\pm^{mnpr}c_{\pm mnqs}=12\delta_{[qs]}^{[pr]}-4c_{\pm\ qs}^{pr}\,.\end{aligned} \tag{8}$$

The tensor $c_\pm^{mnpq}$ transforms as one of the three **35** representations of SO(8). Under the standard SO(7) subgroup of SO(8) defined by the embedding

$$\mathbf{8}_v\to\mathbf{7}+\mathbf{1},\quad \mathbf{8}_\pm\to\mathbf{8}\,, \tag{9}$$

with $\mathbf{8}_\pm$ the two spinor representations of SO(8), the tensors $c_\pm^{mnpq}$ have no invariant part. However, we can utilize the fact that SO(8) has two other SO(7) subgroups, denoted by $\mathrm{SO}(7)^\pm$ and related to the usual one by triality under which $c_\pm^{mnpq}$ does contain a singlet. This is most easily seen using the decomposition for $\mathrm{SO}(8)\supset\mathrm{SO}(7)^+$ with

$$\mathbf{8}_v\to\mathbf{8},\quad \mathbf{8}_+\to\mathbf{7}+\mathbf{1},\quad \mathbf{8}_-\to\mathbf{8}\,, \tag{10}$$

and the corresponding decomposition for $\mathrm{SO}(7)^-$. In our solution, which is invariant under $\mathrm{SO}(7)^\pm$ rotations, the singlet appearing in the decomposition of $\mathbf{8}_\pm$ will correspond to the one unbroken supersymmetry.

The generators of $\mathrm{SO}(7)^\pm$, $G_\pm^{mn}$, can be written as linear combinations of SO(8) generators Γ^{mn} as

$$G_\pm^{mn}=P_{\pm\ pq}^{mn}\Gamma^{pq}\,, \tag{11}$$

with the projection operators $P_\pm$ given by

$$P_{\pm\ pq}^{mn}=\tfrac{3}{4}\left(\delta_{[pq]}^{[mn]}+\tfrac{1}{6}c_{\pm\ pq}^{mn}\right). \tag{12}$$

We thus have

$$P_{\pm\ pq}^{mn}\Gamma^{pq}\eta_\pm=0\,. \tag{13}$$

Finally, following Ref. 5 we consider antisymmetric tensors $F_{\mu\nu}$ obeying the relation

$$F_{\mu\nu}=\lambda_\pm c_{\pm\ \mu\nu}^{\lambda\rho}F_{\lambda\rho}\,, \tag{14}$$

where henceforth $\mu,\nu=1\dots8$ $(m,n=1\dots8)$ are world (tangent space) indices in the eight dimensions transverse to the string. Multiplying both sides of the equation by $c_\pm^{\mu\nu\alpha\beta}$ we deduce that the possible eigenvalues are

$$\lambda_\pm=\tfrac{1}{2},-\tfrac{1}{6}\,. \tag{15}$$

Choosing the eigenvalue $\lambda_\pm=\frac{1}{2}$ corresponds to setting the **7** part of $F_{\mu\nu}$ to zero under the decomposition $\mathbf{28}\to\mathbf{21}+\mathbf{7}$ of the adjoint of SO(8) under either SO(7) subgroup and ensures that $F_{\mu\nu}\gamma^{\mu\nu}\in\mathrm{SO}(7)^\pm$.

Our strategy for constructing a solution preserving precisely one supersymmetry and invariant under $\mathrm{SO}(7)^\pm$ is rather simple. We must construct an ansatz so that the gauge field strength and generalized connection Ω are in a $\mathrm{SO}(7)^\pm$ subgroup of the transverse tangent space SO(8) and so that the dilatino variation vanishes for $\epsilon=\eta_\pm$ using (13). Then we want

$$\Omega_\mu{}^{mn}=\tfrac{1}{2}c_{\pm\ pq}^{mn}\Omega_\mu{}^{pq},\quad F_{\mu\nu}=\tfrac{1}{2}c_{\pm\ \mu\nu}^{\lambda\rho}F_{\lambda\rho}\,. \tag{16}$$

The first condition is satisfied by the choice

$$H_{\mu\nu\lambda}=\tfrac{1}{7}c^\rho_{\pm\ \mu\nu\lambda}\partial_\rho\phi,\quad g_{\mu\nu}=e^{6\phi/7}\delta_{\mu\nu}\,. \tag{17}$$

The connection is then

$$\Omega_\mu{}^{mn}\Gamma_{mn}=\tfrac{1}{7}c^\rho_{\pm\ \mu\nu\lambda}\partial_\rho\phi\,\gamma^{\nu\lambda}-\tfrac{6}{7}\partial_\nu\phi\,\gamma_\mu{}^\nu\,. \tag{18}$$

This implies that the gravitino and dilatino terms in (3) vanish for $\epsilon=\eta_\pm$ for any choice of $\phi(x^\mu)$.

In order to fully determine the solution it remains to solve $F_{\mu\nu}=\frac{1}{2}c_{\pm\ \mu\nu}^{\lambda\rho}F_{\lambda\rho}$ and then to solve the Bianchi identity $dH=(\alpha'/30)\mathrm{Tr}F\wedge F$ for H and ϕ. Remarkably, the equation for $F_{\mu\nu}$ has already been solved in the literature for gauge group SO(7) embedded in the nonstandard $\mathrm{SO}(7)^+$ subgroup of the SO(8) Euclidean rotation group. In order to utilize the solution of Refs. 6 and 7 we first pick an SO(7) subgroup of SO(32) or $\mathrm{E}_8\otimes\mathrm{E}_8$. For SO(32) we choose the canonical $\mathrm{SO}(24)\otimes\mathrm{SO}(8)$ subgroup of SU(32) and then choose a $\mathrm{SO}(7)^+$ subgroup of SO(8) which gives the decomposition

$$\mathbf{496}\to(\mathbf{276},\mathbf{1})\oplus(\mathbf{1},\mathbf{21})\oplus(\mathbf{1},\mathbf{7})\oplus(\mathbf{24},\mathbf{8}) \tag{19}$$

under $\mathrm{SO}(32)\supset\mathrm{SO}(24)\otimes\mathrm{SO}(7)^+$. For E_8 we choose the $\mathrm{SO}(8)\otimes\mathrm{SO}(8)$ subgroup of $\mathrm{SO}(16)\subset\mathrm{E}_8$ and embed $\mathrm{SO}(7)^+$ in one of the SO(8) factors to obtain

$$\begin{aligned}\mathbf{248}\to&(\mathbf{28},\mathbf{1})\oplus(\mathbf{1},\mathbf{21})\oplus(\mathbf{1},\mathbf{7})\oplus(\mathbf{8}_v,\mathbf{8})\\&\oplus(\mathbf{8}_+,\mathbf{1})\oplus(\mathbf{8}_+,\mathbf{7})\oplus(\mathbf{8}_-,\mathbf{8})\end{aligned} \tag{20}$$

under $\mathrm{E}_8\supset\mathrm{SO}(8)\otimes\mathrm{SO}(7)^+$. In either case we have

$\mathrm{Tr}F^2=30\,\mathrm{tr}_8F^2$ with Tr indicating the trace in the adjoint of SO(32) or E_8 and tr_8 the trace in the **8** spinor of SO(7).

The solution of Refs. 6 and 7 is then given by

$$A_\mu^\pm=-\frac{2}{3}\frac{1}{r^2+\rho^2}G_{\mu\nu}^\pm x^\nu\,, \qquad (21)$$

with ρ an arbitrary scale size. The Bianchi identity then gives

$$e^{(6/7)\phi}=e^{(6/7)\phi_0}+\alpha'\frac{1}{3}\frac{2\rho^2+r^2}{(\rho^2+r^2)^2}\,. \qquad (22)$$

Although the form of this solution is reminiscent of SU(2) instantons in four dimensions, there are important differences. In particular, the leading term at large r in (21) is not pure gauge. As a result, the field strength falls off only as $1/r^2$ and the solution is not characterized by an element of $\pi_7(G)$ with $G=\mathrm{SO}(32)$ or $E_8\otimes E_8$.

Although the metric is asymptotically flat, the slow $(1/r^2)$ falloff of the fields implies that the Arnowitt-Deser-Misner expression for the mass per unit length of this string diverges. As for, e.g., axion strings in four dimensions, this divergent energy is an infrared phenomenon, and should not prevent the existence of a well-behaved low-energy effective action governing the string dynamics on scales large relative to its core size. In this action the octonion string has zero thickness and acts as a source for massless spacetime fields.

The construction of this action is a fascinating problem for future research, but we would like to make several preliminary comments. The string world-sheet action has one massless field for every zero mode of the soliton solution. There are, of course, the usual eight zero modes from broken translation invariance. Furthermore, since the solution is not rotationally invariant there are seven rotational zero modes which parametrize an $\mathrm{SO}(8)/\mathrm{SO}(7)^+\sim S^7$ coset space. There are also zero modes arising from gauge rotations and dilation of the string core.

The existence of one unbroken spacetime supersymmetry in the presence of the soliton implies that the world-sheet action has (0,1) supersymmetry.[13] The fifteen broken supersymmetries lead to fifteen world-sheet Goldstinos. The integral of $\mathrm{Tr}F^4$ over the eight transverse dimensions is nonzero for the soliton, which may imply the existence of additional world-sheet fermions via the Atiyah-Singer index theorem. The nonvanishing of this integral has an additional important consequence. Anomaly cancellation requires the term $\int B\wedge \mathrm{tr}F^4$ to appear in the spacetime action. This implies that $B_{\mu\nu}$ must couple to the low-energy action for octonion strings (at sub-leading order in the string loop expansion), and that octonion strings carry the same type of axion charge as fundamental strings. In addition, this octonion string appears to carry a tensor charge which can be measured by embedding it in a 5-brane.[7]

It is also of interest to understand the coupling of the octonionic world sheet to spacetime fields. At the linearized level, these "vertex operators" can be determined by studying the dynamics of octonion strings in spacetime backgrounds with weakly excited plane waves. One such vertex operator must exist for every massless spacetime field of heterotic string theory. This suggests, along with the conjectures of Refs. 9 and 14, the possibility of a direct relationship between octonion and fundamental strings, as yet to be understood.

In this paper, we have only shown that the octonionic soliton is a solution to leading order in α' or heterotic string theory. However, in Ref. 15 low-energy supersymmetry is used to argue that the leading-order heterotic soliton solution of Ref. 9 is the leading approximation to an exact solution of string theory. It would be of interest to see if the arguments of Ref. 15 could also apply to the octonionic soliton.

It is clear that the solution presented here is a rather unique object that circumvents much of the conventional wisdom regarding the behavior of solitons in supersymmetric theories. For example, it preserves only one-sixteenth, rather than one-half, of the supersymmetries. On general grounds, there should exist a Lorentz-covariant supersymmetric action which describes the octonion string propagating in ten dimensions. This action might provide a starting point for an alternative quantization of superstring theory, although it is far from clear whether or not the octonion string corresponds to one of the known fundamental strings.

While this work was in progress, we received an interesting and related paper by Duff and Lu[16] concerning string soliton solutions of a proposed 5-brane effective field theory that utilizes the instanton of Ref. 4.

We thank C. Callan for useful discussions. This work was supported in part by NSF Presidential Young Investigator Grant No. PHY80-19754, by DOE Outstanding Junior Investigator Grant No. 8-48061-2500-3, and by the A. P. Sloan Foundation.

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