Scale-Independent Fluctuations of Spin Stiffness in the Heisenberg Model and Its Relationship to Universal Conductance Fluctuations

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(Received 12 September 1990)

It is shown that, at low temperatures, the absolute thermal fluctuations of the spin-stiffness constant in the classical Heisenberg model are independent of the scale L for $d<4$, if L is larger than the lattice spacing but smaller than the correlation length, while the relative fluctuation is proportional to L^{4-2d}, where d is the dimensionality, and is thus scale independent in $d=2$ (with logarithmic corrections). The phenomenon is strikingly similar to the universal conductance fluctuations known for the problem of an electron in a random potential.

PACS numbers: 75.10.Jm, 72.10.Bg

In 1985, Lee and Stone[1] and Altshuler[2] predicted that the fluctuation of the conductance in a mesoscopic system should be universal (since this prediction the subject has grown immensely.[3]) A sample is considered to be mesoscopic if its linear dimension L is larger than the mean free path l, which acts as a microscopic cutoff in any field-theoretical discussion, but smaller than the length scale at which the phase coherence of electrons is broken. It was shown that if we denote the average conductance by $\langle G\rangle$, and the absolute fluctuation of the conductance by $\langle(\delta G)^2\rangle$, then $\langle(\delta G)^2\rangle$ is independent of scale and is universal for $d<4$, where d is the dimensionality of the sample. The average conductance $\langle G\rangle$ is, of course, proportional to L^{d-2}, with logarithmic corrections in $d=2$. Thus, the relative fluctuation of the conductance, $\langle(\delta G)^2\rangle/\langle G\rangle^2$, is proportional to L^{4-2d} and is scale independent in $d=2$. More recently, Altshuler, Kravstov, and Lerner[4] (AKL) have shown how this result, among many others, can be obtained from a replica field theory. This replica field theory is an extended nonlinear σ model defined on a symplectic Grassmannian manifold. This raises the tantalizing possibility that a similar result should also hold for a much simpler manifold, namely, the coset space of $O(N)/O(N-1)$. In the present paper I shall show that this is indeed so. For $N=3$ this is simply the O(3) nonlinear σ model which faithfully reproduces the long-distance properties of the classical Heisenberg model defined by unit vector spins on a hypercubic lattice.[5]

Because the classical Heisenberg model is so *physically* different from the problem of an electron moving in a random potential, it is important to define carefully the sense in which the result known for the electronic problem generalizes. It follows from AKL that the conductance is a stiffness in response to a twist in the boundary conditions of the replica fields. Thus, the conductance is a susceptibility defined with respect to the twist. Similarly, fluctuations of the conductance and its higher moments, as described by AKL, are nonlinear susceptibilities. This is not such an unusual definition as it may sound. It was pointed out by Kohn[6] that the zero-frequency limit of the imaginary part of the conductivity of a sample can be related to the response, or more precisely to the shift of the energy levels, of the electronic system with respect to a twist in the boundary condition of the electronic wave function. A twisted boundary condition was also beautifully utilized by Edwards and Thouless[7] in the context of Anderson localization. As stated above, the response to the twisted boundary condition, as applied to the electronic wave function, is related to the imaginary, rather than the real, part of the conductivity. However, with some plausible assumptions about the frequency dependence, the response can also be shown to yield the real part.[7] The twist of the boundary condition of the *replica fields*, however, does correctly give the conductance, and it is this definition of the conductance that generalizes to the magnetic problem. For a Heisenberg magnet, defined on a hypercube of linear dimension L, I apply a twist of the spin field on the boundary, and ask, how does the system respond? The response defines the well-known spin-stiffness constant. I then calculate the thermal fluctuation of the spin-stiffness constant and show that the absolute fluctuation is scale independent for $d<4$, while the relative fluctuation is proportional to L^{4-2d} and is therefore scale independent in $d=2$, with logarithmic corrections.

I still need to discuss what are meant by fluctuations in these two disparate physical problems. Clearly, the fluctuations of the conductance of a sample are due to disorder, while in referring to the Heisenberg model we are speaking about a pure system without disorder. However, to the extent that the replica field theory is a meaningful description of the conductance problem, the fluctuations are given by the action of the replica field theory which acts like the Boltzmann weight. The inverse conductivity, *defined on the scale of the mean free path,* then acts as the "temperature." In this description, in which disorder is integrated out at the very beginning, disorder no longer appears explicitly. Thus, the fluctuations due to disorder, as viewed from the replica field theory, are analogous to thermal fluctuations. Of course, in the magnetic problem the fluctuations are nothing but the thermal fluctuations. To clarify this point, consider, for example, the $d=2$ Heisenberg model which does not have long-range order at any finite temperature but has exponentially growing correlation length, $\xi \propto \exp(2\pi\rho_s^0/T)$, for temperatures T low compared to the microscopic spin-stiffness constant ρ_s^0.[5] The macroscopic spin stiffness is, of course, zero. However, at low temperatures, and on length scales smaller than the correlation length, local spin stiffness is a nonvanishing quantity. Thus, the picture is that of locally ordered regions. The fluctuations of the Heisenberg model discussed here are the thermal fluctuations on the scale of these locally ordered regions. Because the correlation length diverges rapidly as the temperature tends to zero, these local regions can be quite large. It should also be noted that the theory of conductance fluctuations, as we know today, only applies to the regime of weak disorder where the localization in $d=2$ is large but finite (see, however, AKL). I shall show that for length scales much larger than the lattice spacing, but much smaller than the correlation length, the absolute fluctuation of the spin-stiffness constant is independent of scale in $d<4$. The average spin-stiffness constant, however, depends logarithmically on the scale in $d=2$, as in the conductance problem.

Let us begin by defining the O(N)-invariant nonlinear σ model which is given by the action

$$S=\frac{\rho_s^0}{2T}\int d^dx\,(\partial_i\hat{\mathbf{\Omega}})^2\,, \tag{1}$$

where the sum over the repeated index i is implied. The N-component vector $\hat{\mathbf{\Omega}}$ has unit magnitude. The partition function Z is given by the sum over all unit-vector spin configurations weighted by the Boltzmann factor e^{-S}. Note that ρ_s^0/T has the dimension Λ^{d-2}, where Λ^{-1} is the microscopic length scale at which it is defined. In particular, it is dimensionless at $d=2$. This model describes a theory of interacting spin waves, similar to the theory of interacting diffusion modes of the replica field theory for the problem of electrons in a random potential. It is, by now, well understood[5] that this theory has asymptotic freedom in $d=2$, i.e., the interactions between the spin waves at short length scales are negligibly small but grow with increasing length scales. Thus, the spin waves propagate freely at short length scales. We also have the same picture in the conductance problem in which the interactions between the short-wavelength diffusion modes are negligibly small.

Now consider a hypercube of linear dimension L, and apply the following boundary conditions in the x direction:[8]

$$\hat{\mathbf{\Omega}}(x=0)=\hat{\mathbf{\Omega}}_1,\quad \hat{\mathbf{\Omega}}(x=L)=\hat{\mathbf{\Omega}}_2\,, \tag{2}$$

where $\hat{\mathbf{\Omega}}_1$ and $\hat{\mathbf{\Omega}}_2$ are two constant vectors making an angle θ, $\hat{\mathbf{\Omega}}_1\cdot\hat{\mathbf{\Omega}}_2=\cos\theta$. Choose now a reference frame such that $\hat{\mathbf{\Omega}}_1=(1,0,0,\ldots)$ and $\hat{\mathbf{\Omega}}_2=(\cos\theta,\sin\theta,0,0,\ldots)$. Now make the following transformation:

$$\hat{\mathbf{\Omega}}=\begin{pmatrix}\cos(\theta x/L) & \sin(\theta x/L) & 0\\ -\sin(\theta x/L) & \cos(\theta x/L) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_1\\ \sigma_2\\ \boldsymbol{\pi}\end{pmatrix}. \tag{3}$$

The boundary conditions satisfied by the new fields are $\sigma_1=1$, $\sigma_2=0$, and $\boldsymbol{\pi}=0$ at $x=0$ and $x=L$. We can choose periodic boundary conditions in the remaining $d-1$ directions. The vector $\boldsymbol{\pi}$ has $N-2$ components, and we must have $\sigma_1^2+\sigma_2^2+\boldsymbol{\pi}^2=1$. Because this transformation is orthogonal, the Jacobian of the measure of the path integral is unity. The new action S is now given by

$$S=\frac{\rho_s^0}{2T}\int d^dx\left[\frac{\theta^2}{L^2}(\sigma_1^2+\sigma_2^2)+(\partial_i\sigma_1)^2+(\partial_i\sigma_2)^2+(\partial_i\boldsymbol{\pi})^2\right]. \tag{4}$$

For convenience, define $h\equiv\theta^2/L^2$, and note that by taking derivatives of $-\ln Z$ with respect to h one can obtain any moment of the spin-stiffness constant. In particular, the first derivative with respect to h defines the spin-stiffness constant.

I shall discuss the renormalization-group analysis of this problem, much along the lines given by AKL, elsewhere. In the present paper I shall only derive the one-loop result which is exactly on the same footing as the original derivative of the conductance fluctuation. To derive the *one-loop* result it is sufficient to keep terms only to quadratic order in the action S. This is because each power of the field variable brings in a factor $(T/\rho_s^0)^{1/2}$ in a loop-wise expansion. After eliminating σ_1, in terms of σ_2 and $\boldsymbol{\pi}$, the quadratic part of the action, S_{qua}, is

$$S_{\text{qua}}=\left(\frac{\rho_s^0}{2T}\right)hL^d+\frac{\rho_s^0}{2T}\int d^dx\,[-h\boldsymbol{\pi}^2+(\partial_i\sigma_2)^2+(\partial_i\boldsymbol{\pi})^2]\,. \tag{5}$$

Note also that the elimination of σ_1 produces a factor $\prod_x[1-\sigma_2^2(x)-\boldsymbol{\pi}^2(x)]^{-1/2}$ in the measure of the path integral. The integral over σ_2 can be trivially carried out and the result, because it does not depend on h, can be lumped into the measure of the path integral. We therefore get, to one-loop order,

$$\bar{\rho}_s \equiv \left[-\frac{\partial}{\partial\theta^2}\ln Z_{\text{qua}}\right]_{\theta\to 0} = L^{d-2}\left(\frac{\rho_s^0}{2T}\right)\left[1-\frac{1}{L^d}\int d^dx\langle\pi^2(x)\rangle\right] = L^{d-2}\left(\frac{\rho_s^0}{2T}\right)\left[1-\frac{N-2}{L^d}\frac{T}{\rho_s^0}\sum_q{}'\frac{1}{q^2}\right], \quad (6)$$

$$\overline{(\Delta\rho_x)^2} \equiv \left[\left(\frac{\partial}{\partial\theta^2}\right)^2 \ln Z_{\text{qua}}\right]_{\theta\to 0} = \frac{1}{L^4}\left(\frac{\rho_s^0}{2T}\right)^2 \int d^dx\, d^dy \langle[\pi^2(x)-\langle\pi^2(x)\rangle][\pi^2(y)-\langle\pi^2(y)\rangle]\rangle = \frac{N-2}{2L^4}\sum_q{}'\frac{1}{q^4}. \quad (7)$$

The lower limits of the primed wave-vector sums in Eqs. (6) and (7) are assumed to be $1/L$. The averages in Eqs. (6) and (7) were carried out with respect to the quadratic action, Eq. (5), except that the first two terms in the integrand are missing, i.e., the action density is simply $(\partial_i\pi)^2$. Equation (6) simply defines the spin-stiffness constant at the scale L, and Eq. (7), the absolute thermal fluctuation of the spin-stiffness constant at the scale L. Note that this is also a nonlinear susceptibility.[9] It is easy to evaluate Eqs. (6) and (7), and to leading order in L, we get, for $2<d<4$, the relative fluctuation to be

$$\overline{(\Delta\rho_s)^2}/\bar{\rho}_s^2 \propto L^{4-2d}. \quad (8)$$

The absolute fluctuation, however, is scale independent for $d<4$. More explicitly, in the interesting case of $d=2$, we get

$$\frac{\overline{(\Delta\rho_s)^2}}{\bar{\rho}_s^2} = \frac{N-2}{2\pi}\left(\frac{T}{\rho_s^0}\right)^2\left(\frac{1}{1-[(N-2)T/2\pi\rho_s^0]\ln(\Lambda L)}\right)^2 = \frac{2\pi}{(N-2)[\ln(\xi/L)]^2}. \quad (9)$$

Note that we have assumed that $\Lambda^{-1} \ll L \ll \xi$; thus the denominator in Eq. (9) never becomes too small. The one-loop result for ξ is, of course, given by the value of L for which the denominator of Eq. (9) vanishes. The last part of Eq. (9) shows that the result can also be expressed in terms of the physical correlation length, thus eliminating both Λ and T/ρ_s^0 from the equation. Of course, the weak logarithmic dependence on the scale L is also present in the electron problem. In the metallic phase of the electron problem, for dimensions $d>2$, the Josephson length[10] ξ_J separates the long-distance Goldstone behavior from the short-distance critical behavior. The average spin-stiffness constant calculated here agrees with the expression obtained previously using a different method.[11]

In conclusion, I have shown that the universal conductance fluctuation discovered in the context of electrons moving in a random potential has a close analog in the fluctuation of the spin-stiffness constant in the classical Heisenberg model. This appears to be both unexpected and novel. I hope that this analogy may serve to deepen our understanding of this problem. A direct experimental verification of the results derived in the present paper may be difficult but not out of the question; a place to look could be a system composed of fine magnetic grains. As mentioned earlier, a renormalization-group analysis of this problem, much along the lines given by AKL, will be given elsewhere. It is also possible that a better understanding of the nature of the moments of the spin-stiffness constant may resolve the difficulties that have been pointed out for the $2+\epsilon$ expansion of the $O(N)$ nonlinear σ model.[12]

I would like to thank the organizers of the workshop on mesoscopic fluctuations at the Aspen Center for Physics, and in particular, A. MacDonald, for providing a most delightful environment in which this work was conceived. I would also like to thank M. P. Gelfand, S. Kivelson, and J. Rudnick for their comments on the manuscript. This work was supported by the National Science Foundation (Grant No. DMR-89-07664).

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