Finite-Size Scaling of Driven Diffusive Systems: Theory and Monte Carlo Studies

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By identifying anisotropic scaling as the dominant feature of a lattice gas driven by a strong electric field, I obtain finite-size scaling forms by extending exact field-theoretic results. These are tested and supplemented by extensive Monte Carlo simulations in two dimensions. Excellent agreements are found. This resolves the disagreements between simulations and theories on the universality of the related models. Novel features of the finite-size effects of intrinsically anisotropic systems are emphasized.

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In recent years, the study of critical phenomena[1] has been extended to nonequilibrium systems under steady-state conditions.[2-10] One of the central issues is whether the important theoretical concept of universality remains pertinent to a detailed description of the singularities near a *steady-state* critical point, as it does in equilibrium.[1] If universality holds, then simplified and theoretically tractable models can be used to understand and predict properties of much more complex systems belonging to the same universality class.

In this regard, an extensively studied system giving rise to controversies is the Ising lattice gas with attractive interactions, driven by a uniform electric field that maintains a current-carrying steady state.[2] When analyzed analytically in dimension $d=2$ under extreme anisotropic jump rates, mean-field exponents were found.[3] With universality in mind, detailed analyses[4-6] of a closely related continuum model yielded a new stable fixed point in an $\epsilon=(5-d)$ expansion.[5,6] Critical exponents were obtained[5] to all orders for $2<d<5$. These models are known as "driven diffusive systems."[11] Despite considerable effort,[2,7-9] however, simulations have only been able to provide qualitative comparisons, and in some cases disagreements, with theories. As an example of the latter, the estimate of the order-parameter exponent[8,9] $\beta\approx 0.23$ in $d=2$ is at odds with $\frac{1}{2}$ as predicted by theories[3,5,6] (subject to possible logarithmic corrections). Moreover, the data seem to be consistent with one correlation-length exponent,[8] rather than two as required by intrinsic anisotropies.

These discrepancies call into question the applicability of the concept of universality to steady-state criticality, casting doubts on the theoretical approach of calculations using simplified models, based on criteria such as symmetries and conservation laws.[12] In view of the considerable interest in nonequilibrium steady states in recent years,[10] especially when most of these systems are confined to be finite on computers, it is not only of broad fundamental interest, but also of urgency, to make detailed comparisons between model calculations and simulations in order to settle this uncertainty. Motivated by this goal, I consider in this Letter the problem of finite-size effects in driven diffusive systems.

For nonequilibrium systems, there is very little exploitation of the idea of finite-size scaling[13] in analyzing simulation data (except some speculations *at* T_c by Binder and Wang[14]). Previous simulations therefore rely on direct estimates,[2,7] isotropic finite-size scaling, or questionable extrapolation procedures to infinite sizes.[8] The origin of the subtlety of finite-size effects is clearly the intrinsic anisotropies (not removable by rescaling of lengths), typical in systems driven by unidirectional forces. To see this systematically, I consider, in the framework of the continuum model, the connected equal-time correlation functions in momentum space

$$G_c^{(N)}(\{k_i\})=\langle\phi(k_1)\cdots\phi(k_N)\rangle_c \Big/ \delta^d\left(\sum_i k_i\right),$$

where $\langle\cdots\rangle_c$ denotes an ensemble average of the cumulant, and ϕ is the local magnetization density. For $2<d<5$, the dangerously irrelevant ϕ^4-coupling constant[5,6] u effectively introduces a new length scale into the problem, and leads to the violation of hyperscaling as in equilibrium.[14-16] From the Callan-Symanzik (CS) equation,[17] we deduce[5] at small k

$$G_c(k)\equiv G_c^{(2)}(k,-k)$$
$$=k_\perp^{-2+\eta_\perp}\tilde{G}(tk_\perp^{-2},k_\parallel k_\perp^{-\lambda},ut^\theta), \qquad (1)$$

where $k_\perp$ ($k_\parallel$) is the component transverse (parallel) to the driving field, and $t=(T-T_c)/T_c$, with T_c the critical temperature of an infinite system. The exponents are exact:[5] $\eta_\perp=0$, $\lambda=2+\epsilon/3$ ($=\nu_\parallel/\nu_\perp$ in Ref. 5), and $\theta=1-\epsilon/3$. The variable $k_\parallel k_\perp^{-\lambda}$ implies that in a renormalization-group (RG) transformation $k_\perp\to k_\perp b$, $k_\parallel\to k_\parallel b^\lambda$ under a scale change by a factor b, in order to keep the dynamic functional (or the Langevin equation) invariant. To generalize Eq. (1) to finite systems of geometry $L_\perp^{d-1}L_\parallel$, we may, according to the RG derivations of finite-size scaling for equilibrium systems,[3,16,18] treat $L_\perp^{-1}$ and $L_\parallel^{-1}$ as two extra (relevant) variables. Linearizing about the fixed point yields

$$G_c(k_\perp,k_\parallel,t,u,L_\perp^{-1},L_\parallel^{-1})=b^2G_c(k_\perp b,k_\parallel b^\lambda,tb^2,ub^{-2\theta},L_\perp^{-1}b,L_\parallel^{-1}b^\lambda)=L_\parallel^{2/\lambda}G_c(k_\perp L_\parallel^{1/\lambda},k_\parallel L_\parallel,tL_\parallel^{2/\lambda},uL_\parallel^{-2\theta/\lambda},S,\text{const}), \quad (2)$$

after setting $b = \text{const} \times L_\parallel^{1/\lambda}$. Equation (1) is recovered under the thermodynamic limit $L_\perp \to \infty$, $L_\parallel \to \infty$ with the shape factor $S \equiv L_\perp^{-1} L_\parallel^{1/\lambda}$ held *fixed*, expecting as usual that $tL_\parallel^{2/\lambda}$ controls the singular behavior of G_c. Hence the exact exponents from field theory[5] (e.g., $\beta = \frac{1}{2}$) are defined in this limit that *does not* preserve the geometry.[19]

For simplicity, I consider henceforth $G^{(N)}(k_m)$ with $k_m \equiv (k_\perp^{(1)} = 2\pi/L_\perp,\ k_\perp^{(2)} = \cdots = k_\perp^{(d-1)} = k_\parallel = 0)$, as motivated by the anisotropic ordering found in simulations.[2,7] Because of conservation, k_m corresponds to $k=0$ for nonconserved systems. Thus

$$G_c(k_m, t, u, L_\perp^{-1}, L_\parallel^{-1}) = L_\parallel^{2/\lambda} \tilde{G}_2(tL_\parallel^{2/\lambda}, U, S) \tag{3}$$

defines $\tilde{G}_2$, where $U \equiv uL_\parallel^{-2\theta/\lambda}$. In equilibrium, the dangerous nature of u leads to the well-known mechanism of multiplicative singularities above the upper critical dimension.[15,16] Thus we expect as $U \to 0$,

$$G_c = L_\parallel^{2/\lambda} U^{\bar{u}_1} \hat{G}_2(tL_\parallel^{2/\lambda} U^{\bar{u}_2}, SU^{\bar{u}_3}) . \tag{4}$$

Apart from the above thermodynamic limit with S fixed, I also consider the limit $L_\perp \sim L_\parallel \to \infty$, as common in most simulation studies. Note that now $S \to 0$, which for *isotropic* systems corresponds to $L_\perp \to \infty$, $L_\parallel \to \infty$ but $L_\parallel / L_\perp \to 0$; to my knowledge, the associated behavior has not been understood. The discussion for the anisotropic case is therefore somewhat speculative. Clearly, S cannot be simply set to zero. Motivated by the ubiquity of power laws near criticality,[19] I conjecture that asymptotically for small S,

$$\begin{aligned} G_c &= L_\parallel^{2/\lambda} U^{\bar{u}_1} (SU^{\bar{u}_3})^p \bar{G}_2(\dot{t} L_\parallel^{2/\lambda} U^{\bar{u}_2} (SU^{\bar{u}_3})^q) \\ &= L_\parallel^{2/\lambda} S^p U^{u_1} \bar{G}_2(\dot{t} L_\parallel^{2/\lambda} S^q U^{u_2}) , \end{aligned} \tag{5}$$

where $u_1 = \bar{u}_1 + \bar{u}_3 p$, $u_2 = \bar{u}_2 + \bar{u}_3 q$, and $\dot{t} \equiv [T - T_c(L_\perp, L_\parallel)]/T_c(L_\perp, L_\parallel)$ takes care of the shift of T_c due to the second variable of $\hat{G}_2$. The form of Eq. (5) is adopted as a working hypothesis, and is of course subject to more rigorous examination such as Ref. 18.

It is important to distinguish between $\dot{t}$ and t as, following the well-known argument,[13]

$$t^* \equiv [T_c(L_\perp, L_\parallel) - T_c]/T_c \approx L_\parallel^{-2/\lambda} U^{-\bar{u}_2} F(SU^{\bar{u}_3}) \tag{6}$$

with some suitable function F. So, for $x \equiv SU^{\bar{u}_3}$, $tL_\parallel^{2/\lambda} S^q U^{u_2} - \dot{t} L_\parallel^{2/\lambda} S^q U^{u_2} \approx x^q F(x)$, instead of a constant as for isotropic block systems. Thus $\dot{t}$, not t, enters Eq. (5).[13]

Similarly, solving the CS equations and assuming the same mechanism for singularities as in Eq. (5) yield

$$m = L_\parallel^{-(1+\theta)/\lambda} S^r U^{u_3} \bar{m}(\dot{t} L_\parallel^{2/\lambda} S^q U^{u_2}) , \tag{7}$$

$$G_c^{(4)}(k_m) = L_\parallel^{(3+d+\lambda)/\lambda} S^s U^{u_4} \bar{G}_4(\dot{t} L_\parallel^{2/\lambda} S^q U^{u_2}) , \tag{8}$$

where m is the order parameter. Since they are derived from the common dynamic generating functional ($\mathcal{F}[J]$, J being an external field)[20] analogous to the free energy in equilibrium,[17] they share the same scaling variable in $\dot{t}$. Furthermore, we expect (see Ref. 16 for details) $u_3 = p_1 - p_2$, $u_1 = p_1 - 2p_2$, and $u_4 = p_1 - 4p_2$, where p_1 and p_2 are associated with the singularities (of U) multiplying $\mathcal{F}$ and J, respectively. The subtractions of p_2 follow from differentiating $\mathcal{F}$ with respect to J. Hence $u_4 = 3u_1 - 2u_3$. Similarly $s = 3p - 2r$.

The relation $\lim_{L \to \infty} \lim_{h \to 0} G_c(k_m)/V = m^2$ is a natural generalization of that for nonconserved systems which has $k=0$ instead of k_m (see Ref. 16 for details). Here h is the magnetic field and V the volume. This gives $u_4 = 4u_3 = 2u_1$, $s = 4r - 3(d-1)$, and $p = 2r - (d-1)$. With these relations, the generalization of the renormalized coupling[21]

$$\begin{aligned} g_L &= -\tfrac{2}{3} G_c^{(4)}(k_m)/V[G_c(k_m)]^2 \\ &= -\tfrac{2}{3} (L_\perp^{-1} L_\parallel^{1/\lambda})^{s-4r+3(d-1)} \bar{G}_4 / \bar{G}_2^2 \\ &\equiv \bar{g}(\dot{t} L_\parallel^{2/\lambda} S^q U^{u_2}) \end{aligned} \tag{9}$$

indicates crossing at $\dot{t} = 0$ for different sizes, since the prefactors cancel out exactly.

Since the u_i should be independent of the value of S, they can be determined by imposing that the L dependence cancel out in each of Eqs. (5), (7), and (8) in the limit $L_\parallel = L_\perp^\lambda \to \infty$, and then identifying the prefactors with those as determined away from T_c by mean-field theory:[6] i.e., $m \propto u^{-1/2}$, $G_c \propto u^0$, and $G_c^{(4)} \propto u$. This procedure has been known to yield hyperscaling violation.[15,16] It leads to $u_1 = u_2 = -\frac{1}{2}$. Unaware of any further theoretical argument and without making any further assumption (e.g., the form of the real-space correlation functions[14]), the two unknown powers (one with S prefactor, another with $\dot{t}$) have to be determined either in a more rigorous approach or by numerical means. Of course, if S is fixed and finite, Eqs. (6)–(9) still hold with the same u_i and with t replacing $\dot{t}$.

In $d=2$, u is marginal.[5,6] Without seeing any reliable way to predict whether it would lead to logarithm corrections in the presence of the relevant variable e that represents the drive, I disregard such corrections in the following discussions. The good fits of the data below support that such corrections are weak and numerically insignificant. Thus, with $\lambda = 3$ and $\theta = 0$, we have from Eq. (7),

$$m = L_\parallel^{-1/3} S^r \bar{m}(\dot{t} L_\parallel^{2/3} S^q) , \tag{10}$$

and likewise for $G_c(k_m)$ and g_L. The shift of T_c is simpler: $t^* = L_\parallel^{-2/3} F(S)$ and hence $tL_\parallel^{2/3} S^q \approx \dot{t} L_\parallel^{2/3} S^q + S^q F(S)$. We now turn to the results of simulations in an attempt to test and supplement these predictions.

Simulation results.— I use the standard Monte Carlo technique[2,22] with Metropolis spin-exchange jump rates on the $d=2$, half-filled, driven Ising lattice gas with attractive interactions, for a wide range of system size, with periodic boundaries. The driving field E is effectively infinite. Most runs are as long as 2×10^6 sweeps to

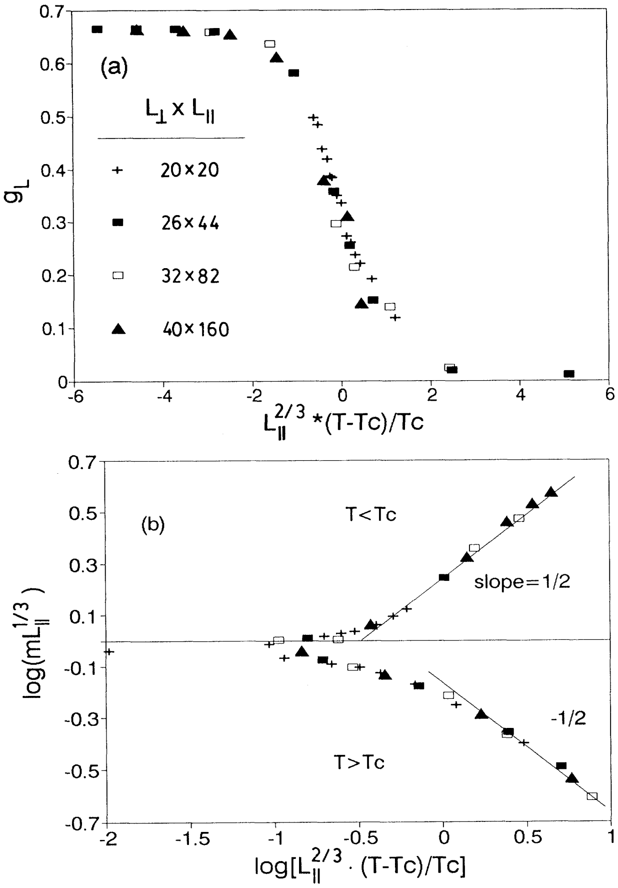

FIG. 1. Scaling plots of fixed-S data, $S=L_\perp^{-1}L_\parallel^{1/3}=0.136$. (a) g_L vs $tL_\parallel^{2/3}$; (b) log-log plot of $mL_\parallel^{1/3}$ vs $tL_\parallel^{2/3}$. The asymptote below T_c shows $\beta=\frac{1}{2}$.

ensure good statistics. Temperature is in units of T_c of the $d=2$ Ising model.

I identify the order parameter as

$$m=\frac{1}{2}\sin\frac{\pi}{L_\perp}\left\langle\left|\sum_{x_\perp}e^{i2\pi x_\perp/L_\perp}\frac{1}{L_\parallel}\sum_{x_\parallel}\sigma(x_\perp,x_\parallel)\right|\right\rangle$$

$$\equiv\langle|\Phi(k_m)|\rangle\,, \qquad (11)$$

where $\sigma=\pm1$ is the local spin variable. The prefactor ensures $m=1$ at $T=0$. Compared to (see Ref. 2 for details) $m'=\langle m_\parallel^2\rangle-\langle m_\perp^2\rangle$, where $\langle m_\parallel^2\rangle$ and $\langle m_\perp^2\rangle$ are the mean-square magnetization of the column and row, m is better for two reasons: First, for $T>T_c$, the finite-size tail $m\sim1/\sqrt{V}$ is much smaller than $1/\sqrt{L}$ of m' (due to the squares *inside* the means); second, m is sensitive to the spatial distribution of the domains, as an order parameter should be, whereas m' is not.[9] The generalized renormalized coupling

$$g_L=\tfrac{4}{3}\left[1-\langle|\Phi(k_m)|^4\rangle/2\langle|\Phi(k_m)|^2\rangle^2\right]$$

is also calculated.[21,23] Above T_c the distribution tends to a Gaussian,

$$\langle|\Phi(k_m)|^4\rangle=\langle\Phi(k_m)^2\Phi(-k_m)^2\rangle\to2\langle|\Phi(k_m)|^2\rangle^2\,,$$

as $L\to\infty$, so $g_L\to0$. Below T_c, $g_L\to\frac{2}{3}$ [Fig. 1(a)].

For runs of *fixed* S, the predictions of Eqs. (9) and (10) (with t replacing i) are clearly borne out, with $T_c\approx1.418\pm0.005$, and $\bar{g}(t=0,\ S=0.136)\approx0.31\pm0.03$, which, though unavailable analytically, differs distinctly from 0.611 of the $d=2$ Ising model in square geometry.[24] Figure 1(b) shows full accord of the data with $\beta=\frac{1}{2}$, and with the high-T tail that implies an amplitude $\propto S^{1/2}$: $m\sim L_\parallel^{-1/3}(L_\parallel^{2/3})^{-1/2}S^{1/2}\propto1/\sqrt{V}$. For runs of *various small* S, I consider if (9) and (10) give good fits. I start with $g_L=\bar{g}(iL_\parallel^{2/3}S^q)$ because it involves q only. The predicted crossing at $i=0$ is confirmed and its scaling behavior is obtained for $q=-\frac{1}{2}$, yielding a scaling variable $i(L_\perp L_\parallel)^{1/2}$. The value $\bar{g}(i=0)\approx0.37\pm0.02$ differs from $\bar{g}(t=0)$ above, due to the S dependence. m in the form of (10) scales excellently with $r=0$ and $T_c(L_\perp,L_\parallel)$ as deduced from g_L, as shown in Fig. 2, which also shows that $\beta_{\rm eff}\approx\frac{1}{3}$ as if $\bar{m}(x)\sim x^{1/3}$ for $x\to\infty$ and $T<T_c$, canceling the L prefactor for $L_\perp\sim L_\parallel\to\infty$. However, plotting the data as in Fig. 1(b) also shows consistency with $\beta=\frac{1}{2}$ for different S, but each with a different curve. Thus, it is likely that $\beta_{\rm eff}$ is an effective exponent describing the crossover to $S=0$, since the data may not have sufficiently small S to reveal the asymptotic behavior of $\bar{m}$. Here a study of the geometry $\infty\times L_\parallel$ may be helpful.[19]

For $T_c(L_\perp,L_\parallel)$, $L_\parallel^{2/3}t^*$ fits $F(S)\sim S^{-0.85}$ quite well, with an extrapolation $T_c(\infty)\approx1.41\pm0.01$ lying well outside the previous estimate[8] 1.355 ± 0.003 that has been used in the literature ever since, but is consistent

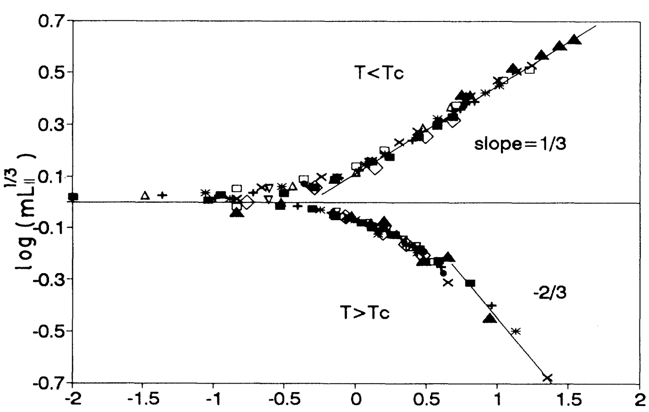

FIG. 2. For different small S, $\log_{10}(mL_\parallel^{1/3})$ vs $\log_{10}[i(L_\perp\times L_\parallel)^{1/2}]$ [cf. Eq. (10)] for square geometry $L\times L$: $L=14$ (■), 20 (+), 30 (*), 40 (□), 50 (×), 100 (▲); and for $L_\perp\times L_\parallel$: 50×30 (●), 32×16 (◇), 16×32 (△), 24×48 (▽). $T_c(L_\perp,L_\parallel)$'s are, respectively, 1.371, 1.373, 1.374, 1.375, 1.376, 1.378, 1.345, 1.33, 1.382, and 1.39.

with my independent estimate from fixed S.

In conclusion, the analysis for fixed S gives strong evidence that the theoretical and simulation models belong to the same universality class. While such an analysis relies heavily on exact field-theoretic results which are not generally available to other anisotropic systems, and that the proposed small-S behavior is phenomenological, it demonstrates clearly that great care must be taken in analyzing data for systems with anisotropic scaling in order to obtain the correct exponents.

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