Critical Phenomena in ^{3}He and ^{4}He at $T=0$ K

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At a critical negative pressure P_c liquid ^{3}He and ^{4}He become macroscopically unstable. We give a theory of the critical behavior near P_c and show that it is in excellent agreement with experiment.

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In recent papers[1,2] we have shown that at a critical negative pressure P_c liquid helium at zero temperature becomes macroscopically unstable. At this pressure the sound velocity goes to zero and the liquid is unstable against long-wavelength density fluctuations. The purpose of this Letter is to present a simple theory of the critical exponents at P_c, and to compare this theory with the experimental data.

The physical origin of the instability at P_c can be understood on the following basis. For any condensed phase at zero temperature the energy E depends on the molar volume V as shown qualitatively in Fig. 1. If a negative pressure is applied, the molar volume increases. However, since dE/dV has a finite maximum value at the inflection point of E vs V, there is an upper limit to the negative pressure that the system can sustain. At this pressure P_c, $d^2E/dV^2=0$ and so the sound velocity vanishes. Near V_c,

$$P-P_c \propto (V-V_c)^2 \tag{1}$$

and the bulk modulus B is proportional to V_c-V. Consequently, the sound velocity c varies as

$$c \propto (V_c-V)^{1/2} \propto (P-P_c)^{1/4}. \tag{2}$$

Of course, the sound velocity near P_c has not been measured. However, for positive pressures, extremely accurate measurements have been made by Abraham *et al.*[3] Analysis of these data by standard techniques[1,2] gives values of P_c of about -9 bars for ^{4}He and -3 bars for ^{3}He.

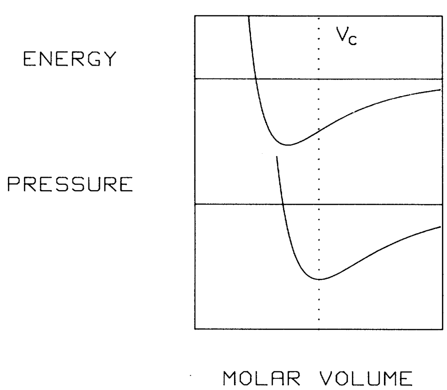

FIG. 1. Qualitative form of the energy and pressure of a liquid as a function of volume. At V_c the pressure takes on its maximum negative value.

The analysis of the sound velocity also gives the critical exponent of c. The result is

$$c \propto (P-P_c)^{\nu}, \tag{3}$$

with ν estimated to be between 0.31 and 0.33 for both ^{3}He and ^{4}He. This value of ν is significantly different from the value $\frac{1}{4}$ obtained in Eq. (2), and this difference is the topic of this Letter.

For c to have the critical behavior (3) the energy near V_c must go as

$$E=E_c+a_1(V_c-V)+a_\gamma(V_c-V)^\gamma+\cdots, \tag{4}$$

where a_1 and a_γ are constants and

$$\gamma=2(1-\nu)/(1-2\nu). \tag{5}$$

Thus, to give $\nu>\frac{1}{4}$, γ must be larger than 3 and so E does not have a simple inflection point.

How can this nonanalytic behavior arise? We suppose that we can divide the energy of liquid helium into two terms, a potential-energy term U, which is taken to be an analytic function of V, and a term K arising from the zero-point energy. Thus,

$$E(V)=U(V)+K(V). \tag{6}$$

For a given $U(V)$ we can find an approximate $K(V)$ by first calculating the bulk modulus $B=Vd^2U/dV^2$. Then, we obtain the sound velocity $c=(B/\rho)^{1/2}$, and next calculate in some approximation the dispersion curve for the elementary excitations. From this dispersion curve we obtain the zero-point energy as the integral over the spectrum, and then find the energy E from Eq. (6).

We can view this calculation as one renormalization cycle. The calculation can now be repeated but using for the bulk modulus the renormalized value $B=Vd^2E/dV^2$. One repeats the cycle until the results do not change, and then examines how the sound velocity varies with volume and pressure near the critical point.

To perform this calculation one needs approximate expressions for the dispersion relation $\epsilon(p)$ and for K. As a simple model we take the energy of the liquid per unit volume to have the form

$$\tfrac{1}{2}\rho v^2+\tfrac{1}{2}B(\delta\rho/\rho)^2+\lambda|\nabla\rho|^2, \quad (7)$$

where v is the liquid velocity, $\delta\rho$ is a change in density from the mean density ρ, and λ is a constant coefficient. Then the dispersion relation is

$$\epsilon=p(c^2+2\lambda\rho p^2/\hbar^2)^{1/2}. \quad (8)$$

The zero-point energy is

$$K=\frac{2\pi V}{\hbar^3}\int_0^{p_m}\epsilon p^2\,dp =\frac{4\pi V}{\hbar^4}\left[\frac{\lambda\rho}{2}\right]^{1/2}p_m^5\left[\frac{2y^5}{15}+\left[\frac{1}{5}-\frac{2y^2}{15}\right](1+y^2)^{3/2}\right], \quad (9)$$

where p_m is a momentum cutoff and $y=\hbar c/p_m(2\lambda\rho)^{1/2}$. With this form for K, we have determined the critical behavior of the sound velocity. It is straightforward to show (either analytically or by numerically performing the iteration process) that near P_c,

$$c\propto(P-P_c)^{1/3}, \quad (10)$$

i.e., $\nu=\frac{1}{3}$ and $\gamma=4$. Thus, the effect of the renormalization process is to eliminate from $E(V)$ the term in $(V_c-V)^3$ that occurs at an ordinary inflection point, and to leave $(V_c-V)^4$ as the first finite term after the linear contribution. This result follows from Eq. (6) if one expands both sides as series in powers of $\tilde{V}\equiv V_c-V$. $U(V)$ contains all powers of $\tilde{V}$, including the square. However, E cannot have a term in $\tilde{V}^2$ because then the sound velocity would not become zero at V_c. Thus, in Eq. (6) the quadratic term in the expansion of U must be balanced by a quadratic term in K, and so the leading term in the sound velocity must go as V_c-V. This then implies that $\gamma=4$.

The result (10) is in *extremely* good agreement with the experimental data. This is shown in Fig. 2 by means of a simple plot of c^3 vs P for ^{3}He and ^{4}He.

The calculation of the zero-point energy does not properly take into account short-wavelength excitations, e.g., rotons. One can make the spectrum (7) more realistic by the addition into (6) of terms involving higher-order gradients of ρ, or terms in $\dot{\rho}^2$, etc. It is straightforward to show that these terms do not affect the critical exponents, at least given the assumption that their coefficients are nonsingular at the critical point. In addition, we have implicitly assumed that K is well approximated by an integral over a spectrum of noninteracting excitations; i.e., anharmonic effects are included in a highly simplified way. We plan to discuss these effects in more detail in a subsequent paper.

If we accept that the variation of c as $(P-P_c)^{1/3}$ is

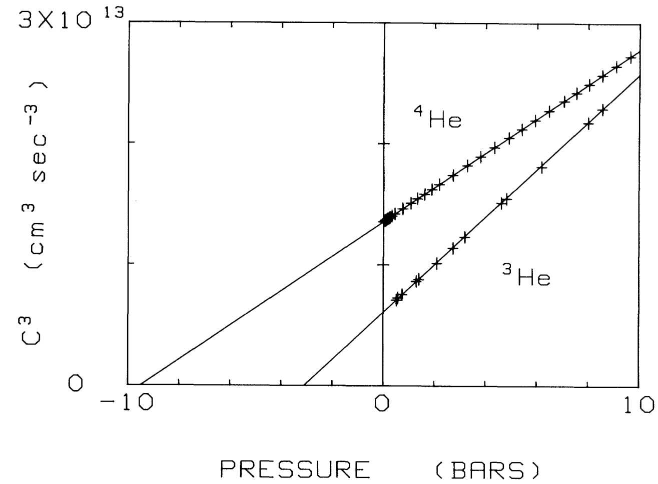

FIG. 2. c^3 vs pressure (c denotes sound velocity) for ^{3}He and ^{4}He. The data are taken from Ref. 3. The theory predicts a linear relation between c^3 and P, when P is not too far from P_c. The two straight lines have been drawn to pass through the first and last data points for ^{3}He and ^{4}He.

indeed an indication of a critical phenomenon, it is surprising that this law holds so well so far from the critical point. The plots in Fig. 2, for example, extend out to values of $1-V/V_c$ of 0.18 and 0.30 for ^{3}He and ^{4}He, respectively. It is clearly unrealistic to try to use the simple theory we have given here to estimate the extent of the critical region, or to calculate the size of the leading correction to the $(P-P_c)^{1/3}$ behavior.

It would be very interesting to test the theory by more accurate experiments. Measurements of c close to P_c would provide the most direct test, but such measurements are extremely difficult to make. A static negative pressure applied to the liquid will lead to the nucleation of bubbles within a short time.[4] As far as we can see, the most promising method for improving the accuracy of the measured critical exponents would be a more accurate study of the sound velocity in ^{3}He as a function of pressure. The data of Ref. 3 give roughly comparable accuracy for ν from ^{3}He and ^{4}He. This is because the ^{4}He data are about 1 order of magnitude more accurate, but the ^{3}He data extend much closer to P_c. If sound-velocity measurements in ^{3}He could be made of comparable accuracy to those made in ^{4}He (fractional errors about $\pm3\times10^{-5}$), one could probably obtain values for ν accurate[5] to about ±0.001. This would provide a rigorous test of the theory we have presented here.

In summary, we have given a simple theory of the critical behavior of liquid helium close to the negative pressure at which it becomes unstable. This theory is in very good agreement with experimental measurements of the sound velocity as a function of pressure.

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[3]B. M. Abraham, Y. Eckstein, J. B. Ketterson, M. Kuchnir, and P. R. Roach, Phys. Rev. A **1**, 250 (1970); B. M. Abraham, D. Chung, Y. Eckstein, J. B. Ketterson, and P. R. Roach, J. Low Temp. Phys. **6**, 521 (1972).

[4]See, for example, P. L. Marston, J. Low Temp. Phys. **25**, 383 (1976).

[5]This is based on a determination of the exponent ν from simulated data with random fractional errors of $\pm 3\times 10^{-5}$ using the same method as in Ref. 2. Thirty data points were used between 0 and 15 bars.